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*S E L E C T P A R T S*  
O F  
S A U N D E R S O N ' S  
E L E M E N T S  
O F  
A L G E B R A .

FOR THE USE OF  
STUDENTS AT THE UNIVERSITIES.

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THE FIFTH EDITION;

REVISED AND CORRECTED

By the Rev. JOHN HELLINS,  
of Trinity College, Cambridge;  
and Vicar of Potter's Pury, Northamptonshire.

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L O N D O N :

Printed for C. DILLY, S. BLADON, W. RICHARDSON,  
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M D C C X C I I .

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## A D V E R T I S E M E N T.

**T**HE Excellence of Professor SAUNDERSON'S *Elements of Algebra* is universally acknowledged: But as that WORK contains many CURIOUS and ELEGANT PIECES, which are rather of Advantage and Amusement to Proficients in the general Science of the Mathematics, than of necessary Use to Students in Algebra; some of the principal Tutors in the University of *Cambridge* were desirous of having such Parts selected from the Whole, as would give their Pupils a clear and comprehensive Knowledge of Algebra, without putting them promiscuously to the Expence of purchasing the original Work, which was published in Two Volumes, Quarto. The Public is indebted to a Gentleman in

that University, of distinguished Judgment and Skill in the Mathematics, for selecting the several Parts which make up the following Work : Nor have any Endeavours of the Editor been wanting, to have it printed as correctly as possible.

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## POSTULATA.

**B**EFORE I enter upon my province, it may not be amiss to acquaint my young disciple what preparations he is to make, and what qualifications I expect of him beforehand, that we may neither of us find ourselves disappointed afterwards. I expect then that he knows how to add, to subtract, to multiply, to divide, to find a fourth proportional, and to extract roots, especially the square root: nay, I expect further, that he shall not only be able to perform all these operations exactly and readily, but also that he shall be able to apply them upon all common occasions; in a word, I expect that he be tolerably well skilled in common Arithmetick, at least so far as relates to whole numbers: for this reason it is that I have prefixed a few arithmetical questions, wherein he may first try his strength and skill before he ventures any further; they are for the most part very easy. I cannot say indeed they are the best chosen, but they were such as lay in my way when I first began this work and was hastening to matters of greater moment; and I do not see but they may, if studied with care and attention, answer well enough the end they were intended for. If he finds no difficulty in these, he will have little reason to doubt of his success afterwards; but if he does, he ought then at last to become sensible of his own defects, and to endeavour to supply whatever is wanting, and to correct whatever is amiss, before he enters himself under my conduct; in the mean time he has my leave to

A

hope

hope that I shall be less upon the reserve with him when he falls more immediately under my care.

*N. B.* The *praxis* of the rule of proportion, and of the rule for extracting the square root, not being (properly speaking) of the nature of simple *postulata*, but rather deducible from the four first; I shall not fail to demonstrate these rules so soon as I shall find proper opportunities for that purpose.

*Questions for exercise in Multiplication.*

Multiplication is taking any one number called the multiplicand as often as is expressed by any other number called the multiplier, and the number produced by this operation is called the product: whence it follows, that the product contains the multiplicand as often as there are units in the multiplier, and that if a number of a greater denomination is to be reduced to an equivalent number of a less, it must be done by multiplication. As for example; In a pound sterling there are 20 shillings; therefore, in every sum of money consisting of even pounds, there are twenty times as many shillings as there are pounds; therefore, if any number of pounds be multiplied by 20, the product will be an equivalent number of shillings; and the same must be observed in all other cases.

QUEST. I.

*It is required to reduce 456 pounds 13 shillings and 4 pence, into shillings, pence, and farthings.*

<i>Answer.</i>	Shillings	9133
	Pence	109600
	Farthings	438400.

# M U L T I P L I C A T I O N .

3

## Q U E S T . 2 .

*A certain island contains 36 counties, every county 37 parishes, every parish 38 families, and every family 39 persons : I demand the number of parishes, families, and persons in the whole island?*

*Answer.* Parishes 1332  
Families 50616  
Persons 1974024

## Q U E S T . 3 .

*In 1730 years, 42 weeks, and 3 days, how many minutes?*

*N. B.* A year consists of 365 days, 6 hours; and an hour of 60 minutes.

Hours in one year	8766
In 1730 years	15165180
In 42 weeks, 3 days	7128
In the whole	15172308
Minutes in the whole	910338480

## Q U E S T . 4 .

*There is a certain field 102004 feet long, and 102003 feet broad : I demand the number of square feet therein contained?*

*Answer.* 10404714012.

## Q U E S T . 5 .

*There is a certain floor 24 feet 4 inches broad, and 96 feet 6 inches long : I demand how many square inches are therein contained?*

*Answer.* 338136 square inches.

## QUESTIONS IN

## QUEST. 6.

*A certain piece of wood 1 foot 2 inches thick, 3 feet 4 inches broad, and 5 feet 6 inches long, is to be cut into small cubes like dies, each of which is to be a quarter of an inch every way: I demand into how many dies the whole may be resolved?*

*Answer.* The whole may be resolved into 2365440 dies.

## QUEST. 7.

*I demand the number of changes that may be rung on 12 bells?*

Changes upon	2 bells	2
on	3 bells	6
on	4 bells	24
on	5 bells	120
on	6 bells	720
on	7 bells	5040
on	8 bells	40320
on	9 bells	362880
on	10 bells	3628800
on	11 bells	39916800
on	12 bells	479001600

If the learner finds himself unprepared to answer the three following questions, he may omit them, and pass on to exercise himself in division.

## QUEST. 8.

*How many different ways can four common dies come up at one throw?*

*Answer.* 1296 ways.

## QUEST. 9.

*Suppose one undertake to throw an ace at one throw with four common dies; what probability is there of his effecting it?*

*Answer.* By the last question four dies can come up 1296 different ways with and without the ace;  
and

and by a like computation, they can come up 625 ways without the ace; therefore there are 671 ways wherein one or more of them may turn up an ace; therefore the undertaker has the better of the lay in the proportion of 671 to 625.

Q U E S T. 10.

*There are two inclosures of the same circumference, that is, both inclosed with the same number of pales; but one is a square whose side is 125 feet, and the other an oblong or long square, 124 feet in breadth, and 126 in length: quære which is the greater close; that is, which, cæteris paribus, will bear most grafs?*

*Answer.* The square: for that contains 15625 square feet; whereas the other contains but 15624.

*Questions for exercise in Division.*

The design of division is, to shew how often one number called the divisor is contained in another called the dividend, and the number that shews this is called the quotient; whence, and from the definition of multiplication already given, I observe, 1<sup>st</sup>, That the divisor multiplied by the quotient, and consequently the quotient multiplied by the divisor, will always be equal to the dividend, provided there be no remainder after the division is over; but if there be, then this remainder added to, or taken into, the product, will give the dividend, which is the best proof of division. 2<sup>dly</sup>, That as the divisor is such a part of the dividend as is expressed by the quotient; so also is the quotient such a part as is expressed by the divisor. Thus 12 divided by 3 quotes 4; therefore 3 is a fourth part, and 4 a third part, of 12. 3<sup>dly</sup>, Hence may a number be found that shall be divisible by any two given numbers whatever without remainders, to wit, by multiplying the two given num-

bers together. Thus if I would have a number that can be divided by both 6 and 9 without any remainders, I multiply 9 by 6, and the product 54 will answer both conditions; though 18 be the least number of that kind. *4thly*, Multiplication and division by the same number are the reverse of each other, and so must necessarily have contrary effects: for whereas multiplication increases a number by taking it as often as is expressed by the multiplier, division (on the contrary) lessens it, by taking only such a part of it as is expressed by the divisor. *5thly*, Hence, if a number of a lesser denomination be to be changed into an equivalent number of a greater, as farthings into pence, pence into shillings, &c. it must be done by division, as the reverse is done by multiplication. *6thly*, Whenever it is proposed to know how often one quantity of any kind is contained in another of the same kind, the numbers representing these quantities must be reduced to the same denomination before any division can take place. Thus if I would know how many thirteen-pence-halfpennies there are in twenty shillings, I must not only reduce the thirteen-pence-halfpenny to 27 halfpence, but also the 20 shillings into 480 halfpence; and then must enquire by division how often 27 halfpence are contained in 480 halfpence, that is, how often 27 is contained in 480; the quotient is 17, and the remainder 21, that is 21 halfpence; for in all division, the remainder must be of the same denomination with the dividend whereof it is a part; therefore in 20 shillings there are 17 thirteen-pence-halfpennies, and 10 pence halfpenny over.

QUEST. II.

*It is required to reduce 987654321 farthings into pounds, shillings, and pence?*

*Answer.* 987654321 farthings are equivalent to 246913580 pence and 1 farthing; or to 20576131 shillings, 8d. 1q.; or to 1028806 pounds, 11s. 8d. 1q.

QUEST.

QUEST. 12.

*One lends me 1296 guineas when they were valued at 1l. 1s. and sixpence a piece: how many must I pay him when they are valued at 1l. 1s. a piece?*

*Answer.* 1326 guineas 18 shillings.

QUEST. 13.

*A certain floor 24 feet 4 inches broad, 96 feet 6 inches long, is to be laid at the rate of 12 pence the square foot: I demand what the whole charge will amount to?*

*Answer.* The floor contains 338136 square inches, or 2348 square feet and 24 square inches; therefore the whole charge amounts to 117 pounds 8 shillings and two pence.

QUEST. 14.

*There is a certain cooler 36 inches deep, 42 inches wide, and 72 inches long: I demand its solid content in English gallons?*

*Note.* An ale gallon is 282 cubic inches.

*Answer.* The vessel contains 108864 cubic inches, that is, 386 gallons, and 12 cubic inches over.

QUEST. 15.

*A cubic foot of water weighs 76 pounds Troy or Roman weight; and the air is 860 times lighter than water: I demand the weight of a cubic foot of air.*

*N. B.* A pound Troy contains 12 ounces, one ounce 20 pennyweights, and one pennyweight 24 grains.

A 4

*Answer.*

*Answer.* A cubic foot of air weighs Troy weight  
1 oz. 1 pwt. 5 gr.

QUEST. 16.

*The mean time of a lunation, that is, from new moon to new moon, is 29 days, 12 hours, 44 minutes, and 3 seconds; and a Julian year consists of 365 days 6 hours: I demand then how many lunations are contained in 19 Julian years.*

Hours in a Lunation	708
Minutes	42524
Seconds	2551443
Hours in 19 Julian years	166554
Minutes	9993240
Seconds	599594400
Lunations 235; and 1 hour 28' 15'' over.	

QUEST. 17.

*In what time may all the changes on 12 bells be rung, allowing 3 seconds to every round? See Question the 7th.*

The number of changes on 12 bells 479001600  
 The time 1437004800 seconds,  
 or 23950080 minutes,  
 or 399168 hours,  
 or 45 years, 27 weeks, 6 days, 18 hours.

QUEST. 18.

*A General of an army distributes 15 pounds 19 shillings and 2 pence halfpenny among 4 captains, 5 lieutenants, and 60 common soldiers, in the manner following: Every captain is to have 3 times as much as a lieutenant, and every lieutenant twice as much as a common soldier: I demand their several shares?*

The

The share of a common soldier	3s. 4d. $\frac{3}{4}$
of a lieutenant	6s. 9d. $\frac{1}{2}$
of a captain	1l. 0s. 4d. $\frac{1}{2}$

*Questions for exercise in the Rule of Three.*

*And first in the Rule of Three Direct.*

The rule of proportion, or rule of three, or by some the golden rule, is that which teacheth, having three numbers given, to find a fourth proportional, that is, to find a fourth number that shall have the same proportion to some one of the numbers given, as is expressed by the other two; and therefore, whenever a question is proposed wherein such a fourth proportional is required, that question is said to belong to the rule of proportion. Now in questions of this nature, especially where the numbers given are not merely abstract numbers, but are applied to particular quantities, three things are usually required, to wit, preparation, disposition, and operation.

First, as to the preparation, it must be observed, that of the three numbers given in the question, two will always be of the same kind, and must be reduced to the same denomination, if they be not so already; and if the remaining number be of a mixt denomination, that also must be reduced to some simple one.

Secondly, in disposing the numbers thus prepared, those two that are of the same denomination must be made the first and third numbers in the rule of proportion, and consequently the remaining number must be the second. But here particular care must be taken, that of the two numbers that are of the same denomination, that be made the third in the rule of proportion, upon which the main stress of the question lies, or to which the question more immediately relates, or which contains the demand: and

the place of this number being once known, the other two must take their places as above directed. This ordering of the numbers for the operation is commonly called stating of the question.

Lastly, having thus stated the question, multiply the second and third numbers together; divide the product by the first, and the quotient thence arising will be the fourth number sought; which fourth number, as well as the remainder, if there be any, must always be understood to be of the same denomination with the second. As for example.

### QUEST. 19.

*A piece of plate weighing 3 pounds 4 ounces and 5 pennyweights, Troy weight, is valued at 5 shillings and 6 pence an ounce; what is the value of the whole?*

Here we have three quantities concerned in the question, viz. 3 pounds 4 ounces and 5 pennyweights; one ounce; and 5 shillings and 6 pence; whereof the two first, which are of the same kind, must be reduced to the same denomination, and the last to a simple one; thus: for one ounce I write 20 pennyweights; for 3 pounds, 4 ounces, and 5 pennyweights, 805 pennyweights; and for 5 shillings and 6 pence, 66 pence; and so the numbers are sufficiently prepared. In the next place I enquire which of the two numbers 20 and 805, which are of the same denomination, is that upon which the main stress of the question lies, and I find it to be 805; for the main business of this question is, to enquire into the value of 805 pennyweights of plate; the rest being no more than *data* in order to discover this: So I make 805 my third number; 20, which is a number of the same denomination, my first; and 66 my second; and state the question thus: *If 20 pennyweights of plate be worth 66 pence, what will 805 pennyweights of plate be*

be

*be worth?* Now to answer this question, I multiply 805 by 66, and the product is 53130; this I divide by my first number 20, and the quotient is 2656, and there remains 10, that is, 10 pence; therefore, to render my quotient more compleat, I bring the remaining 10 pence into 40 farthings, and so divide again by 20, and find the quotient to be 2, that is, 2 farthings, without any remainder; so the value sought is 2656 pence 2 farthings; that is, 11 pounds 1 shilling and 4 pence halfpenny.

*A demonstration of this Praxis.*

*Case 1st.* Now to demonstrate this manner of operation, I shall resume the foregoing question, but at first under a different supposition, as thus: *If one pennyweight of plate cost 66 pence, what will 805 pennyweights cost?* Here nobody doubts but that upon this supposition, 805 pennyweights will cost 805 times 66 pence, or 66 times 805, that is, 53130 pence; therefore in all instances of this kind, that is, where the first number in the rule of proportion is unity, the fourth number must be found by multiplying the second and third numbers together.

*Case 2d.* Let us now put the question as it was at first stated, to wit, *If 20 pennyweights of plate be worth 66 pence, what will 805 pennyweights be worth?* Now upon this supposition it is easy to see, that neither 1 pennyweight, nor consequently 805 pennyweights, will be worth above a 20th part of what they were in the former case; and therefore we must not now say that 805 pennyweights are worth 53130 pence, but a 20th part of that sum, viz. 2656 pence 2 farthings: and as this way of reasoning will be the same in all other instances, it follows now, that *In the rule of proportion, let the numbers given be what they will, the fourth number must be had by multiplying the second and third numbers together, and dividing the product by the first.* Q. E. D.

## QUEST. 20.

*How far will one be able to travel in 7 days 8 hours, at the rate of 13 miles every four hours, allowing 12 hours to a travelling day?*

*Answer.* 299 miles.

## QUEST. 21.

*What will 1296 yards of walling amount to, at the rate of 4 shillings and 5 pence a rod, a rod being 5 yards and a half?*

*Answer.* 52 pounds 8 pence 3 farthings.

## QUEST. 22.

*In the mint of England a pound of gold, that is, 11 ounces fine and 1 alloy, is at this time coined into 44 guineas and an half: I demand how much sterling a pound of pure gold is worth, observing that the alloy is valued at nothing?*

*Answer.* 50 pounds 19 shillings and 5 pence  $\frac{1}{2}$  penny.

## QUEST. 23.

*What is the annual interest of 987 pounds 6 shillings and 5 pence, at the rate of 6 per cent.?*

*Answer.* 59 pounds 4 shillings and 9 pence  $\frac{1}{2}$  penny.

## QUEST. 24.

*The circumference of the Earth, according to the French mensuration, is 123249600 French feet: I demand the same in English miles?*

N. B.

## THE RULE OF THREE. 13

*N. B.* A thousand French feet are equivalent to 1068 English feet; 3 feet make a yard, and 1760 yards make a mile.

*Answer.* 131630573 English feet,  
           or 43876857 yards and 2 feet,  
           or 24930 miles, 57 yards and 2 feet.

### QUEST. 25.

*Supposing all things as in the foregoing question, I demand how long a sound will be in passing from pole to pole, upon a supposition that a sound passes over 1142 feet in a second of time?*

*Answer.* 16 hours and 32 seconds.

### QUEST. 26.

*Monsieur Huygens found that at Paris, the length of a pendulum that swung seconds, was three feet 8 lines and  $\frac{1}{2}$ : I demand its length in English measure?*

*Note.* A line is  $\frac{1}{12}$  part of an inch, and 1000 French half lines are equivalent to 1068 English half lines, as in the 24th question.

*Answer.* The length in English measure of a pendulum that swings seconds, is 94 $\frac{1}{2}$  English half lines; or 39 inches 2 lines and  $\frac{1}{2}$ .

### QUEST. 27.

*I demand in how long a time a pipe, that discharges 15 pints in 2 minutes 34 seconds, will fill a cistern that is 36 inches deep, 42 inches wide, and 72 inches long? (See question the 14.)*

*Answer.* In 31707 seconds; or 8 hours 48' 27".  
For

For as eight pints make 1 gallon, so also eight cubic half inches, that is, eight small cubes of half an inch every way, make one cubic inch; therefore a pint contains 282 cubic half inches, and fifteen pints 4230; but the whole vessel contains 108864 cubic inches by quest. 14; which are equivalent to 870912 cubic half inches; therefore this question ought to be stated thus;

*If 4230 cubic half inches be discharged in 154 seconds of time, in what time will 870912 cubic half inches be discharged? And the answer is,*

In 8 hours 48' 27" as above.

#### QUEST. 28.

*If a wall 6 feet thick, 9 feet high, and 432 feet long, cost 720 pounds in building, what will be the price of a wall of the same materials, that is 12 feet thick, 18 feet high, and 576 feet long?*

In the former wall are contained 23328 cubic feet; in the latter 124416; therefore the answer to this question is 3840 pounds.

#### QUEST. 29.

*A certain steeple projected upon level ground a shadow to the distance of 57 yards, when a four-foot staff perpendicularly erected cast a shadow of 5 feet 6 inches: what was the height of the steeple?*

*Answer.* 41 yards 1 foot 4 inches.

## QUEST. 30.

*Two persons A and B make a joint stock; A puts in 372 pounds, and B 496 pounds, for the same time; and they gain 114 pounds 2 shillings: I demand each man's share of the gain?*

Both their stocks make 868 pounds; say then, if 868 pounds stock bring in 114 pounds 2 shillings gain, what will 372 pounds, *A's* part of the stock, bring in? *Answer.* 48 pounds 18 shillings for *A's* share of the gain; and this subtracted from the whole gain, leaves 65 pounds 4 shillings for *B's* share of the gain.

*Note.* If there be ever so many partners, their shares of the gain must all be found by the rule of proportion, except the last, which may be had by subtraction; but it would be better to find them all by the rule of proportion, because then, if all the shares, when added together, make up the whole gain, it will be an argument that the work is rightly performed.

## QUEST. 31.

*Two persons A and B make a joint stock; A puts in 496 pounds for 2 months, and B 620 pounds for 3 months; and they gain 456 pounds: What will be each man's share of the gain?*

In order to give an answer to this question, it must be considered, that it is the same in the case of trade, as it is in that of money let out to interest, where time is as good as money, that is, whoever lets out 496 pounds for 2 months, is intitled to the same share of the whole gain, as if he had let out twice as much, that is 992 pounds, for one month: in like manner, he that lets out 620 pounds for 3 months, has a right to the same share of the gain,

as if he had let out three times as much, that is, 1860 pounds, for one month: substitute therefore these suppositions instead of those in the question, which may safely be done without affecting the conclusion, and then this question will be reduced to the form of the last, without any consideration of the particular quantity of time, thus; *Two merchants A and B make a joint stock; A puts in 992 pounds, and B 1860 pounds for the same time; and they gain 456 pounds: What will be their respective shares of the gain?*

*Answer.* A's share will be 158 pounds 12 shillings and 2 pence; and B's, 297 pounds 7 shillings and 10 pence.

### QUEST. 32.

*If two men in three days will earn 4 shillings, how much will 5 men earn in six days?*

This and the following question belong to that which they call the double rule of three, wherein 5 numbers are concerned: these numbers must always be placed as they are in this example, that is, the two last numbers must always be of the same denomination with the two first respectively, and the number sought of the same denomination with the middle one; then may the question be reduced to the single rule of three two ways, either by expunging the first and fourth numbers, or the second and fifth. If you would have the first and fourth numbers expunged, you must argue thus; two men will earn as much in three days, as one man in two times 3, or 6 days; also 5 men will earn as much in 6 days, as one man in thirty days; substitute therefore this supposition and this demand, instead of those in the question; and it will stand thus; *If one man in 6 days will earn four shillings, how much*

much will one man earn in 30 days? Which is as much as to say, *If in 6 days a man will earn 4 shillings, how much will he earn in 30 days?*

*Answer.* 20 shillings.

If you would have the second and fifth numbers expunged, you must argue thus: Two men will earn as much in three days, as 3 times two or 6 men in one day; also 5 men will earn as much in 6 days, as 30 men in one day; put then the question this way, and it will stand thus: If 6 men in one day will earn 4 shillings, how much will 30 men earn in one day? That is, *If in any quantity of time 6 men will earn 4 shillings, how much will 30 men earn in the same time?*

*Answer.* 20 shillings, as before.

Whosoever attends to both these methods of extermination, will easily fall into a third, which includes both the other, and in practice is much better than either of them; for at the conclusion of both operations, the number sought was found by multiplying 30 by 4, and dividing the product by 6: Now if he looks back, and traces out these numbers, he will find that the number 30 came from the multiplication of the two last numbers 5 and 6 together, that 4 was the middle number in the question, and that the divisor 6 was the product of the two first numbers 2 and 3 multiplied together; therefore, *In all questions of this nature, if the three last numbers be multiplied together, and the product be divided by the product of the two first, the quotient will give the number sought, without any further trouble.*

## QUEST. 33.

*If for the carriage of three hundred weight 40 miles, I must pay 7 shillings and 6 pence, what must I pay for the carriage of five hundred weight 60 miles?*

*Answer.* 225 pence, or 18 shillings and 9 pence.

*Questions in the Rule of Three Inverse.*

Hitherto we have instanced in the rule of three direct: but there is also another rule of proportion, called the rule of three inverse; which, as to the preparation and disposition of its numbers, differs nothing from the rule of three direct, but only in the operation; for whereas there, the fourth number was found, by multiplying the second and third numbers together, and dividing by the first; here it is found by multiplying the first and second numbers together, and dividing by the third. All that remains then, is to be able to distinguish, when a question belongs to one rule, and when to the other; in order to which, observe the following directions: *If more requires more, or less requires less, work by the rule of three direct; but if more requires less, or less requires more, work by the rule of three inverse.* The meaning whereof is, that if, when the third number is greater than the first, the fourth must be proportionably greater than the second; or if, when the third number is less than the first, the fourth must be proportionably less than the second, the question then belongs to the rule of three direct: But if, when the third number is greater than the first, the fourth must be less than the second; or when the third number is less than the first, the fourth must be greater than the second; in either of these cases, the question belongs to the rule of three inverse, and must be resolved as above directed.

As for example,

QUEST. 34.

*If 12 men eat up a quantity of provision in 15 days, how long will 20 men be in eating the same?*

This question is of such a nature, that more requires less; for 20 men will consume the same provision in less time than 12; therefore the question belongs to the rule of three inverse; so I multiply the first and second numbers together, and divide by the third, and the quotient 9, that is, 9 days, is an answer to the question.

*A demonstration of the Rule of Three Inverse.*

If I was to answer this question by pure dint of thought, without any rule to direct me, I should reason thus: whatever quantity of provision lasts 12 men 15 days, the same will last 1 man 12 times as long, that is, 12 times 15, or 180 days; but if it will last 1 man 180 days, it will last 20 men but the 20th part of that time, that is, 9 days: here then the fourth number was found by multiplying the first and second numbers together; and dividing the product by the third; and the reason is the same in all other cases, where-ever the rule of three inverse is concerned. Q. E. D.

QUEST. 35.

*One lends me 372 pounds for 7 years and 8 months, or 92 months: how long must I lend him 496 pounds for an equivalent?*

*Answer.* 5 years, 9 months.

## QUEST. 36.

*If a square pipe, 4 inches and 5 lines wide, will discharge a certain quantity of water in one hour's time; in what time will another square pipe, 1 inch and 2 lines wide, discharge the same quantity from the same current?*

The orifice of a square pipe 4 inches 5 lines, or 53 lines wide, contains 2809 square lines; and the orifice of a pipe 1 inch 2 lines, or 14 lines wide, contains 196 square lines. Say then, *If an orifice of 2809 square lines will discharge a certain quantity of water in one hour; in what time will an orifice of 196 square lines discharge the same?*

*Answer.* In 14 hours 19' 54<sup>h</sup>.

## QUEST. 37.

*If 3 men, or 4 women, will do a piece of work in 56 days, how long will 1 man and 1 woman be in doing the same?*

Because of the 3 men, or 4 women, some number must be found that is divisible both by 3 and by 4 without remainder; such an one is the number 12, which is the product of 3 and 4 multiplied together; (see observation the third upon the definition of division :) make then 3 men or 4 women equivalent to 12 boys, and you will have 1 man equivalent to 4 boys, 1 woman to 3 boys, and 1 man and 1 woman to 7 boys, and the question will stand thus: *If 12 boys will do a piece of work in 56 days, how long will 7 boys be in doing the same?*

*Answer.* 96 days.

## QUEST. 38.

*If 5 oxen, or 7 colts, will eat up a clofe in 87 days, in what time will 2 oxen and 3 colts eat up the same?*

*Answer.* In 105 days.

## QUEST. 39.

*If 2 acres of land will maintain 3 horses 4 days, how long will 5 acres maintain 6 horses?*

This question may perhaps, at first sight, be taken to be somewhat of the same nature with the 32d and 33d questions, which belonged to the double rule of three direct; but when it comes to be examined into more narrowly, it will be found to be of a very different nature; for we cannot say here as we did there, that 2 acres will last 3 horses as long as 1 acre will last 6 horses; this would be a very unjust way of thinking, and where-ever it is so, the question ought to be referred to another rule, which they call the double rule of three inverse; the propriety or impropriety of this thought being an infallible criterion whereby to distinguish when a question belongs to one rule, and when to the other. All questions belonging to this rule, as well as those belonging to the other, may be reduced to the single rule of three two ways; either by expunging the first and fourth numbers, or the second and fifth; but then the methods of extermination are different. In questions of this nature, if the first and fourth numbers are to be expunged, the 2 first numbers are to be multiplied by the fourth, and the 2 last by the first; but if the second and fifth numbers are to be expunged, then the two first numbers are to be multiplied by the fifth, and the two last by the second; thus in the question before us, if we

would exterminate the first and fourth numbers, we must multiply the two first numbers, that is, 2 and 3, by the fourth, that is, by 5, and say, that 2 acres will last three horses just as long as 10 acres will last 15 horses; we must also multiply the 2 last numbers, to wit, 5 and 6, by the first, that is, by 2, and say, that 5 acres will last 6 horses as long as 10 acres will last 12 horses. Use now these numbers instead of those in the question, and it will be changed into this equivalent one; If 10 acres of land will maintain 15 horses 4 days, how long will 10 acres maintain 12 horses? Strike out of the question the first and fourth numbers, which, being equal, will be of no use in the conclusion, and then the question will stand thus: *If 15 horses will eat up a certain piece of ground in 4 days, how long will 12 horses be in eating up the same?*

*Answer.* 5 days; for this question belongs to the rule of three inverse.

If we would exterminate the second and fifth numbers out of the question, we must multiply the two first numbers by the fifth, and say, that 2 acres will last 3 horses just as long as 12 acres will last 18 horses; we must also multiply the two last numbers by the second, and say, that 5 acres will last 6 horses as long as 15 acres will last 18 horses: use these numbers instead of those in the question, and it will be changed into this equivalent one: If 12 acres will maintain 18 horses 4 days, how long will 15 acres maintain 18 horses? That is (striking out the second and fifth numbers), *If 12 acres of land will maintain a certain number of horses 4 days, how long will 15 acres last the same number?*

*Answer.* 5 days as before; for this question belongs to the rule of three direct.

In both these operations the number sought was at last found by multiplying 15 by 4, and then dividing the product by 12: now whosoever looks back upon the foregoing resolution, and observes how these numbers were formed, he will easily perceive, that the number 4 was the middle term in the question; that the number 15 in both operations was the product of the numbers 3 and 5, which lay next the middle term on each side; and that the divisor 12 was in both cases the product of the extreme numbers 2 and 6: therefore, *In all questions belonging to the double rule of three inverse, where the numbers are supposed to be ordered as in the double rule of three direct, if the three middle numbers be multiplied together, and the product be divided by the product of the two extremes, the quotient of this division will be the number sought.* And thus may all the trouble of expunging be avoided, though I thought it proper to explain that method in the first place, in order to let the learner into the reason of this last theorem, which is founded upon it.

*Questions wherein the Extraction of the Square Root is concerned.*

#### QUEST. 40.

*There is a certain field, whose breadth is 576 yards, and whose length is 1296 yards: I demand the side of a square field equal to it?*

*Answer.* This field will be equal to a square whose side is 864 yards.

#### QUEST. 41.

*There is a certain inclosure 3 times as long as it is broad, whose area is 46128 square yards: I demand its breadth and length?*

The breadth multiplied into the length, that is, the breadth multiplied into 3 times itself, is 46128;

B 4

therefore

therefore the breadth multiplied into itself is 15376;  
therefore the breadth is 124, and the length 372.

## QUEST. 42.

*A certain society collect among themselves a sum amounting to 15 pounds 5 shillings and a farthing, every one contributing as many farthings as there were members in the whole society: I demand the number of members?*

*Answer.* 121 members.

# THE INTRODUCTION,

*Concerning Vulgar and Decimal Fractions.*

N. B. In the following pages certain words are printed in *Italics*, which, being collected from their different lines and read together, will be found to convey the main sense of the author; and consequently these are the words most to be noted. The principal rules and propositions are also printed in *Italics*, with a view to assisting such young students as this work was designed for.

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## DEFINITIONS.

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*Art. 1.*     **A** *FRACTION*, simply and abstractedly considered, is that wherein some part or parts of an unit are expressed: as, if an unit be supposed to be divided into 4 equal parts, and three of these parts are to be expressed, it must be done by the fraction three fourths, to be written thus,  $\frac{3}{4}$ : here the number 4, which shews into how many equal parts the unit is supposed to be divided, and so determines the true value, magnitude, or denomination of those parts, is called the denominator of the fraction; and the number 3, which shews how many of these parts are considered in the fraction, is called the numerator: thus in the fraction  $\frac{1}{2}$  or one half, 1 is the numerator, and 2 the denominator: in  $\frac{2}{2}$  or two halves, 2 is both numerator and denominator, &c.

When a fraction is applied to any particular quantity, that quantity is called the integer to the fraction; thus in  $\frac{3}{4}$  of a penny, a penny is the integer;

teger ; in three fourths of six, the number 6 is the integer ; thus *in three fourths of five sixths, the fraction five sixths is the integer* ; for though in an absolute sense it be a fraction, yet here, with respect to the fraction three fourths, it is an integer : and thus may one and the same quantity, under different ways of conception, be both an integer and a fraction ; as a foot is an integer, and a third part of a yard is a fraction, though they both signify the same thing. *When the integer to a fraction is not expressed, unity is always to be understood.* Thus  $\frac{3}{4}$  is  $\frac{3}{4}$  of an unit ; thus when we say \*,  $\frac{1}{3}$  and  $\frac{1}{4}$  make  $\frac{7}{12}$ , the meaning is, that if  $\frac{1}{3}$  part of an unit, and  $\frac{1}{4}$  part of an unit be added together, the sum will amount to the same as if that unit had been divided into 12 equal parts, and 7 of those parts had been taken. Thus again, when we say that  $\frac{2}{3}$  of  $\frac{4}{5}$  are equivalent to  $\frac{8}{15}$ , we mean, that if an unit be divided into 5 equal parts, and 4 of them be taken, and then this fraction  $\frac{4}{5}$  be again divided into 3 equal parts, and two of them be taken, the result will be the same as if the unit had at first been divided into 15 equal parts, and 8 of them had been taken ; and whatever is true in the case of unity, will be equally true in the case of any other integer whatever. Thus if it be true that  $\frac{1}{3}$  and  $\frac{1}{4}$  of an unit are equal to  $\frac{7}{12}$  of an unit, that is, if it be true in general that  $\frac{1}{3}$  and  $\frac{1}{4}$  added together are equal to  $\frac{7}{12}$ , it will be as true of any particular integer, suppose of a pound sterling, that  $\frac{1}{3}$  of a pound, and  $\frac{1}{4}$  of a pound when added together, are equal to  $\frac{7}{12}$  of a pound ; again, if it be true in general that  $\frac{2}{3}$  of  $\frac{4}{5}$  are equal to  $\frac{8}{15}$ , it is as true in particular that  $\frac{2}{3}$  of  $\frac{4}{5}$  of a pound are equivalent to  $\frac{8}{15}$  of a pound, &c.

\* This will be more fully understood by the student, when he shall have hereafter learned the manner of adding fractions together.

*Of Proper and Improper Fractions ; and of the Reduction of an Improper Fraction to a whole or mixt Number.*

2. *Fractions are of two sorts, proper and improper ; a proper fraction is that, whose numerator is less than the denominator, as  $\frac{1}{2}$  ; therefore an improper one is that, whose numerator is equal to, or greater than the denominator, as  $\frac{2}{2}$ ,  $\frac{3}{2}$ , &c.*

O B J E C T I O N.

But is there no absurdity in the supposition of an improper fraction, as in three halves for instance, considering that an unit cannot be divided into more than two halves ? *Answer :* No more than there is in supposing three halfpence to be the price of any thing, considering that a penny cannot be divided into above two halfpence. These *fractions* therefore *are called improper*, not from any absurdity either in the supposition or in the expression, but *because they may be more properly and more intelligibly expressed, either by a whole number, or at least by a mixt number consisting of a whole number and a fraction ; as for example, if the numerator of a fraction be equal to the denominator, as  $\frac{4}{4}$ , that fraction will always be equivalent to unity, as  $\frac{4}{4}$  of an hour, that is, four quarters of an hour, are equivalent to one hour,  $\frac{4}{4}$  of a penny, that is, 4 farthings, are equal to one penny, &c. and the reason is plain ; for if an unit be divided into four equal parts, and four of these parts be expressed in a fraction, the whole unit is expressed in that fraction, that is, such a fraction must always be looked upon as equal to an unit : therefore if the numerator be double of the denominator, as  $\frac{8}{4}$ , the fraction must be equal to the number 2, because  $\frac{8}{4}$  contain  $\frac{4}{4}$  or 1 twice ; in like manner  $\frac{12}{4}$  are equal to,*  
and

and may be more properly expressed by, the number 3;  $\frac{1^6}{4}$  by the number 4, &c. and universally, as often as the numerator of a fraction contains the denominator, so many units is that fraction equivalent to: But to find how often the numerator contains the denominator, is to divide the numerator by the denominator; therefore if the numerator of an improper fraction be divided by the denominator, the quotient, if nothing remains, will be the whole number by which the fraction may be expressed: but if any thing remains of this division, then the quotient, together with a fraction whose numerator is that remainder, and denominator the divisor, will be a mixt number, expressing the fraction proposed. Thus  $\frac{2^4}{3}$  are equivalent to the whole number 8, but  $\frac{2^5}{3}$  are equivalent to the mixt number  $8\frac{1}{3}$ ,  $\frac{2^6}{3}$  to the mixt number  $8\frac{2}{3}$ , just as 24 feet are equal to 8 yards, 25 feet to 8 yards and 1 foot, 26 feet to 8 yards and 2 feet, &c. and this is what we call the reduction of an improper fraction into a whole or mixt number.

*The Reduction of a whole or mixt Number into an Improper Fraction.*

3. As unity may be expressed by any fraction of any form or denomination whatever, provided the numerator be equal to the denominator, as  $\frac{2}{2}$ ,  $\frac{3}{3}$ ,  $\frac{4}{4}$ , &c. so the number 2 is reducible to any fraction whose numerator is double the denominator, as  $\frac{4}{2}$ ,  $\frac{6}{3}$ ,  $\frac{8}{4}$ , &c. and so is every number reducible to any fraction, whose numerator contains the denominator as often as there are units in the number proposed: therefore whenever a whole number is to be reduced to a fraction whose denominator is given, it must be multiplied by that given denominator, and the product with that denominator under it, will be the equivalent fraction. Thus, if the number 5 is to be reduced

reduced into halves, that is, into a fraction whose denominator is 2, it must be multiplied by 2, and so you will have 5 equal to  $\frac{10}{2}$ , just as 5 pence are equivalent to 10 halfpence; if the number 8 is to be reduced into thirds, it must be multiplied by 3, and so you will have 8 equal to  $\frac{24}{3}$ , just as 8 yards are equal to 24 feet; lastly, if the number 2 is to be reduced into fourths, it will be equal to  $\frac{8}{4}$ , just as 2 pence are equal to 8 farthings. *If the number to be reduced be a mixt number, consisting of a whole number and a fraction, the whole number must always be reduced to the same denomination with the fraction annexed, and the rule will be this: Multiply the whole number by the denominator of the fraction annexed; add the numerator to the product, and the sum with the denominator under it will be the equivalent fraction.* Thus the mixt number  $5\frac{1}{2}$  is equivalent to  $\frac{11}{2}$ , just as 5 pence halfpenny in money is equivalent to 11 halfpence: This operation carries its own evidence along with it; for the number 5 itself is equal to  $\frac{10}{2}$  as above; therefore  $5\frac{1}{2}$  must be equivalent to  $\frac{11}{2}$ : again, the number  $8\frac{2}{3}$  is equal to  $\frac{26}{3}$ , just as 8 yards and 2 feet over are equivalent to 26 feet; lastly,  $2\frac{3}{4}$  is reducible to  $\frac{11}{4}$ , just as two pence and 3 farthings are reducible to 11 farthings.

## A L E M M A.

4. *If any integer be assumed, as a pound sterling, and also any fraction, as  $\frac{3}{4}$ , I say then, that  $\frac{3}{4}$  parts of one pound amount to the same as  $\frac{1}{4}$  part of 3 pounds.*

To demonstrate this Lemma, (which scarcely wants a demonstration) I argue thus: If any quantity, greater or less, be always divided into the same number of parts, the greater or less the quantity so divided is, the greater or less will the parts be. Thus  $\frac{1}{4}$  of a yard is 3 times as much as  $\frac{1}{4}$  of a foot; because a yard is 3 times as much as a foot; and for the



1 penny and 3 farthings. Lastly, suppose I would know the amount of  $\frac{2}{3}$  parts of 17 shillings and 6 pence, I should argue thus:  $\frac{2}{3}$  parts of 17 shillings and 6 pence are equivalent to  $\frac{1}{3}$  part of twice as much, that is, to  $\frac{1}{3}$  part of 35 shillings: but  $\frac{1}{3}$  part of 35 shillings is 11 shillings and 8 pence; therefore  $\frac{2}{3}$  parts of 17 shillings and 6 pence make 11 shillings and 8 pence.

Of the reverse of this reduction, one single instance will suffice: Let it then be required to reduce 1 shilling 2 pence 3 farthings to fractional parts of a pound: here I consider, that in 1 pound are 960 farthings; and in 1 shilling 2 pence 3 farthings, are 59 farthings; therefore 1 farthing is  $\frac{1}{960}$  of a pound; and 1 shilling 2 pence 3 farthings are  $\frac{59}{960}$  of a pound.

*Preparations for further Reductions and Operations of Fractions.*

6. All the operations and reductions of fractions are mediately or immediately deducible from the following principle; which is, that *If the numerator of a fraction be encreased, whilst the denominator continues the same, the value of the fraction will be encreased proportionably: and vice versâ. On the other hand, if the denominator be encreased in any proportion, whilst the numerator continues the same, the value of the fraction will be diminished in a contrary proportion; and vice versâ.* Thus  $\frac{2}{3}$  are twice as much as  $\frac{1}{3}$ , and  $\frac{1}{6}$  is but half as much as  $\frac{1}{3}$ .

From this principle it follows, that *if the numerator and denominator of a fraction be both multiplied, or both divided, by the same number, the value of the fraction will not be affected thereby; because, as much as the fraction is encreased by multiplying the numerator, just so much again it will be diminished by multiplying the denominator: and as much as the fraction is diminished by divid-*  
ing

ing the numerator, just so much again it will be encreased by dividing the denominator. Thus the terms of the fraction  $\frac{3}{4}$  being doubled, produce  $\frac{6}{8}$ , a fraction of the same value; and, on the contrary, the terms of the fraction  $\frac{6}{8}$  being halved, give  $\frac{3}{4}$ .

Hence it appears, that *every fraction is capable of infinite variety of expression*, since there is infinite choice of multipliers, whereby the numerator and denominator of a fraction may be multiplied, and so the expression may be changed, without changing the value of the fraction. Thus the fraction  $\frac{1}{2}$ , if both the numerator and denominator be multiplied by 2, becomes  $\frac{2}{4}$ ; if by 3,  $\frac{3}{6}$ ; if by 4,  $\frac{4}{8}$ ; if by 5,  $\frac{5}{10}$ ; and so on *ad infinitum*; all which are nothing else but different expressions of the same fraction: therefore, in the midst of so much variety, we must not expect that every fraction we meet with should always be in its least or lowest terms; but how to reduce them to this state whenever they happen to be otherwise, shall be the business of the next article.

*The reduction of Fractions from higher to lower Terms.*

7. *Whenever a fraction is suspected not to be in its least terms, find out, if possible, some number that will divide both the numerator and denominator of the fraction without any remainder; for if such a number can be found, and the division be made, the two quotients thence arising will exhibit respectively the numerator and denominator of a fraction, equal to the fraction first proposed, but expressed in more simple terms: this is evident from the last article. As for example: let the fraction  $\frac{10}{15}$  be proposed to be reduced: here, to find some number that will divide both the numbers 10 and 15 without any remainder, I begin with the number 2, as being the first whole number that can have any effect*

effect in division; but I find 2 will not divide 15: 3 is the next number to be tried; but neither will that succeed, for it will not divide 10; as for the number 4, I pass that by, because if 2 would not divide 15, much less will 4 do it; the next number I try is 5, and that succeeds; for if 10 and 15 be divided by 5, the quotients will be 2 and 3 respectively, each without remainder; therefore the fraction  $\frac{10}{15}$ , after being reduced to its least terms, is found to be the same as  $\frac{2}{3}$ ; that is, if an unit be divided into 15 equal parts, and ten of them be taken, the amount will be the same as if it had been divided into 3 equal parts, and 2 of them had been taken. Secondly, if the fraction proposed to be reduced be  $\frac{2520}{7560}$  divide its terms by 2, and you will have the fraction  $\frac{1260}{3780}$ ; divide again by 2, and you will have  $\frac{630}{1890}$ ; divide again by 2, and you will have  $\frac{315}{945}$ ; therefore all further division by 2 is excluded; divide then these last terms by 3, and you will have  $\frac{105}{315}$ ; divide again by 3, and you will have  $\frac{35}{105}$ ; divide by 5, and you will have  $\frac{7}{21}$ ; and lastly divide by 7, and you will have  $\frac{1}{3}$ ; so that the fraction  $\frac{2520}{7560}$ , after a common division by 2, 2, 2, 3, 3, 5, 7, is found at last equal to  $\frac{1}{3}$ . Thirdly, the fraction  $\frac{36}{48}$ , after a continual division by 2, 2, 3, becomes  $\frac{3}{4}$ . Fourthly,  $\frac{56}{84}$ , after a continual division by 2, 2, 7, becomes  $\frac{2}{3}$ . Fifthly,  $\frac{144}{180}$ , after a continual division by 2, 2, 3, 3, becomes  $\frac{4}{5}$ . Sixthly,  $\frac{42}{126}$ , after a continual division by 2, 3, and 7, becomes  $\frac{1}{3}$ . Seventhly,  $\frac{315}{840}$ , after a

continual division by 3, 5, 7, becomes  $\frac{3}{8}$ . Eighthly,  $\frac{35}{840}$ , after a continual division by 5 and 7, becomes  $\frac{1}{24}$ . Ninthly,  $\frac{735}{245}$ , after a continual division by 5, 7, 7, becomes  $\frac{3}{1}$ , or 3.

Some perhaps may think themselves helped in the practice of this rule by the following observations:

First, that 2 will divide any number that ends with an even number, or with a cypher, as 36, 30, &c. and no other.

Secondly, that 5 will divide any number that ends with a 5, or with a cypher, as 75, 70, &c. and no other.

Thirdly, that 3 will divide any number, when it will divide the sum of its digits added together: thus 3 will divide 471, because it will divide the number 12, which is the sum of the numbers 4, 7, and 1.

Fourthly, if both the numerator and denominator have cyphers annexed to them, throw away as many as are common to both: thus  $\frac{3500}{56000}$  is the same as  $\frac{35}{560}$ , or  $\frac{7}{112}$ , or  $\frac{1}{16}$ .

After all, there is a certain and infallible rule for finding the greatest common divisor of any two numbers whatever, that have one, whereby a fraction may be reduced to its least terms by one single operation only. I shall be forced indeed to postpone the demonstration of this rule to a more convenient place, not so much for want of principles to proceed upon, as for want of a proper notation; but the rule itself is as follows: Let  $a$  and  $b$  be two given numbers, whose greatest common divisor is required; to wit,  $a$  the greater, and  $b$  the less: then, dividing  $a$  by  $b$  without any regard to the quotient, call the remainder  $c$ ; divide again  $b$  by  $c$ , and call the remainder  $d$ ; then

then divide  $c$  by  $d$ , and call the remainder  $e$ : then divide  $d$  by  $e$ , and call the remainder  $f$ ; and so proceed on, till at last you come to some divisor, as  $f$ , which will divide the preceding number  $e$  without a remainder: I say then, that this last divisor will be the greatest common divisor of the two given numbers  $a$  and  $b$ . As for example; let  $a$  be 1344 and  $b$  582: then, to find the greatest common divisor of these numbers, I divide  $a$  (1344) by  $b$  (582) and there remains 180, which I call  $c$ ; then I divide  $b$  (582) by  $c$  (180) and there remains 52, which I call  $d$ ; then I divide  $c$  (180) by  $d$  (42) and there remains 12, which I call  $e$ ; then I divide  $d$  (42) by  $e$  (12) and there remains 6, which I call  $f$ ; lastly I divide  $e$  (12) by  $f$  (6) and there remains nothing: whence I conclude that 6 is the greatest common divisor of the two numbers 1344 and 582: and as the quotients by 6 are 224 and 97, it follows, that the fraction  $\frac{582}{1344}$ , when reduced to its least terms, will

be  $\frac{97}{224}$ . If no common divisor can be found but unity, it is an argument that the fraction is in its least terms already.

From this and the last article it follows, that *all fractions that are reducible to the same least terms are equal*; as  $\frac{4}{6}$ ,  $\frac{6}{9}$ ,  $\frac{10}{15}$ , &c. which are all reducible to  $\frac{2}{3}$ ; though it does not follow *à converso*, that *all equal fractions are reducible to the same least terms*; this will be demonstrated in another place. (See *Elements of Algebra*, Art. 193. page 299, 4to.)

For the better understanding of the following article, it must be observed, that this mark  $\times$  is a sign of multiplication, and is usually read *into*: thus  $2 \times 3$  signifies 6,  $2 \times 3 \times 4$  signifies 24,  $2 \times 3 \times 4 \times 5$  signifies 120, &c.; and in some cases it will be better to put down these components or factors, than the character of the number arising from their continual multiplication, as in the following article. It ought

also to be observed, that it matters not in what order these components are placed; for  $2 \times 3 \times 4 \times 5$  signifies just the same as  $4 \times 5 \times 2 \times 3$ , &c.

*The Reduction of Fractions of different Denominations to others of the same Denomination.*

8. There is another reduction of fractions, no less useful than the former; and that is, the reduction of fractions of different denominations to others of the same denomination, or which have the same denominator, without changing their values; which is done as follows: Having first put down the fractions to be reduced, in any order, one after another, and *beginning with the numerator of the first fraction, multiply it, by a continual multiplication, into all the denominators but its own*, and put down the product under that fraction; then multiply, in like manner, the numerator of the next fraction into all the denominators but its own, and put down the product under that fraction; and so proceed on through all the numerators, always taking care to except the denominator of that fraction whose numerator is multiplied. Then, *multiplying all the denominators together*, put down the product under every one of the products last found, and you will have a new set of fractions, all of the same denomination with one another, and all of the same values with their respective original ones. As for example; let it be proposed to reduce the following fractions to the same denomination,  $\frac{1}{2}$ ,  $\frac{3}{4}$ ,  $\frac{5}{6}$ ,  $\frac{7}{8}$ : 1<sup>st</sup>, The numerator of the first fraction is 1, and the denominators of the rest are, 4, 6, and 8, and  $1 \times 4 \times 6 \times 8$  gives 192; therefore I put down 192 under  $\frac{1}{2}$ . 2<sup>dly</sup>, The numerator of the second fraction is 3, and the denominators of the rest are 6, 8, and 2, and  $3 \times 6 \times 8 \times 2$  gives 288; therefore I put down 288 under  $\frac{3}{4}$ . 3<sup>dly</sup>,  $5 \times 8 \times 2 \times 4$  gives 320; therefore I put down 320 under  $\frac{5}{6}$ . 4<sup>th</sup>,  $7 \times 2 \times 4 \times 6$  gives 336; therefore

fore I put down 336 under  $\frac{7}{8}$ . Lastly,  $2 \times 4 \times 6 \times 8$ , or the product of all the denominators, is 384. This therefore I put down under every one of the numerators last found, and so have a new set of fractions, viz.  $\frac{192}{384}$ ,  $\frac{288}{384}$ ,  $\frac{320}{384}$ ,  $\frac{336}{384}$ , all of the same denomination, as appears from the operation itself; and all of the same value with their respective original ones, as will appear presently; but first see the work:

$$\begin{array}{cccc} \frac{1}{2}. & \frac{3}{4}. & \frac{5}{6}. & \frac{7}{8}. \\ \frac{192}{384}. & \frac{288}{384}. & \frac{320}{384}. & \frac{336}{384}. \end{array}$$

*A Demonstration of the Rule.*

All that is to be demonstrated in this rule is, to prove from the nature of the operation itself, that the original fractions suffer nothing in their values by this reduction: in order to which, it will be convenient to put down the components of the new numerators instead of their proper characters, as in the last article; as also those of the common denominator, and the work will stand thus:

$$\begin{array}{cccc} \frac{1}{2}. & \frac{3}{4}. & \frac{5}{6}. & \frac{7}{8}. \\ \frac{1 \times 4 \times 6 \times 8}{2 \times 4 \times 6 \times 8}. & \frac{3 \times 6 \times 8 \times 2}{4 \times 6 \times 8 \times 2}. & \frac{5 \times 8 \times 2 \times 4}{6 \times 8 \times 2 \times 4}. & \frac{7 \times 2 \times 4 \times 6}{8 \times 2 \times 4 \times 6}. \end{array}$$

By this method of operation it appears, that the numerator and denominator of the first fraction  $\frac{1}{2}$ , are both multiplied by the same number in the reduction, to wit, by  $4 \times 6 \times 8$ ; and therefore that fraction suffers nothing in its value, by art. 6. In like manner, the terms of the second fraction  $\frac{3}{4}$  are both multiplied by the same number  $6 \times 8 \times 2$ ; therefore that fraction can suffer nothing in its value; and the same may be said of all the rest. Q. E. D.

*Other Examples to this Rule.*

$$\begin{array}{cccccc}
 \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\
 \frac{360}{720} & \frac{240}{720} & \frac{180}{720} & \frac{144}{720} & \frac{120}{720} & \frac{120}{360} & \frac{90}{360} & \frac{72}{360} & \frac{60}{360} \\
 \\
 \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{5} & \frac{1}{6} \\
 \frac{30}{120} & \frac{24}{120} & \frac{20}{120} & \frac{6}{30} & \frac{5}{30}
 \end{array}$$

The use of this rule will soon appear in the addition and subtraction of fractions: in the mean time it may not be amiss to observe, that *it would be very difficult, if not impossible, to compare fractions of different denominations, without first reducing them to the same.* As for instance; suppose it should be asked, which of these two fractions is the greater;  $\frac{3}{4}$ , or  $\frac{5}{7}$ ; in this view it would be difficult to determine the question; but when I know that  $\frac{3}{4}$  are the same with  $\frac{21}{28}$ , and that  $\frac{5}{7}$  are the same with  $\frac{20}{28}$ , I know then that  $\frac{3}{4}$  are greater than  $\frac{5}{7}$  by a twenty-eighth part of the whole. We now proceed to the four operations of fractions, to wit, their addition, subtraction, multiplication, and division: and first,

*Of the Addition of Fractions.*

9. *Whenever two or more fractions are to be added together, let them first be reduced to the same denomination, if they be not so already; and then, adding the new numerators together, put down the sum with the common denominator under it. In the case of mixt numbers, add first the fractions together, and then the whole numbers: but if the fractions, when added together, make an improper fraction, reduce it by the 2d art. to a whole or mixt number; and then putting down the fractional part, if there be any, reserve the whole number for the place of integers.*

To

To this rule might be referred (if it had not been taught already in the 3d art.) the reduction of a mixt number into an improper fraction, which is nothing else but adding a whole number and a fraction together, and may be done by *considering the whole number as a fraction whose denominator is unity.*

*Examples of Addition of Fractions.*

1<sup>st</sup>,  $\frac{3}{4}$  and  $\frac{4}{8}$  when added together make  $\frac{7}{8}$ , for just the same reason as 3 shillings and 4 shillings when added together make 7 shillings.

2<sup>dly</sup>, The fractions  $\frac{1}{3}$  and  $\frac{1}{4}$  when reduced to the same denomination by the last art. are  $\frac{4}{12}$  and  $\frac{3}{12}$ , and these added together make  $\frac{7}{12}$ ; therefore the fractions  $\frac{1}{3}$  and  $\frac{1}{4}$  when added together make up the fraction  $\frac{7}{12}$ .

For a better confirmation of these abstract conclusions, but chiefly to inure the learner to conceive and reason distinctly about fractions, it may be very convenient to apply these examples in some particular case; as for instance, in the case of a pound sterling; and if we do so here, we are to try, whether  $\frac{1}{3}$  and  $\frac{1}{4}$  of a pound, when added together, amount to  $\frac{7}{12}$  of a pound, or not: here then we shall find by division, that the third part of a pound is 6 shillings and 8 pence, and the fourth part 5 shillings; and these, added together, make 11 shillings and 8 pence; therefore  $\frac{1}{3}$  and  $\frac{1}{4}$  of a pound, when added together, make 11 shillings and 8 pence; but by the 5th art. it will be found that  $\frac{7}{12}$  of a pound are also 11 shillings and 8 pence; therefore  $\frac{1}{3}$  and  $\frac{1}{4}$  of a pound, when added together, make  $\frac{7}{12}$  of a pound; and the same would have been true in any other instance whatever.

3<sup>dly</sup>,  $\frac{2}{5}$  and  $\frac{3}{8}$ , that is,  $\frac{1}{4}\frac{6}{10}$  and  $\frac{1}{4}\frac{5}{10}$ , when added together, make  $\frac{3}{4}\frac{11}{10}$ , which will also be true in the case of a pound sterling; for by the 5th art.  $\frac{2}{5}$  of a pound are 8 shillings,  $\frac{3}{8}$  of a pound are 7 shillings and 6 pence, and their sum is 15 shillings and 6 pence; which will also be found to be the value of

$\frac{3}{4}$  of a pound; therefore  $\frac{2}{3}$  and  $\frac{3}{8}$  of a pound, when added together, make  $\frac{17}{24}$  of a pound.

4thly,  $\frac{2}{3}$  and  $\frac{4}{5}$ , that is,  $\frac{10}{15}$  and  $\frac{8}{15}$ , when added together, make  $\frac{18}{15}$ , an improper fraction; which being reduced to a mixt number, by the 2d art. is 1 and  $\frac{3}{5}$ : let us now try, whether  $\frac{2}{3}$  of a pound, and  $\frac{4}{5}$  of a pound, when added together, will make one pound and  $\frac{4}{5}$  of a pound over, or not: now  $\frac{2}{3}$  of a pound, or 13 shillings and 4 pence, added to  $\frac{4}{5}$  of a pound, or 16 shillings, amount to 1 pound 9 shillings and 4 pence: and  $\frac{4}{5}$  of a pound are found to be 9 shillings and 4 pence; therefore  $\frac{2}{3}$  and  $\frac{4}{5}$  of a pound, when added together, make one pound and  $\frac{4}{5}$  of a pound over.

5thly,  $\frac{3}{4}$  and  $\frac{5}{6}$ , that is,  $\frac{15}{24}$  and  $\frac{20}{24}$ , when added together, make  $\frac{35}{24}$ , or 1  $\frac{11}{24}$ , which will also be true in the case of a pound sterling.

6thly,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ ,  $\frac{1}{6}$ , that is,  $\frac{360}{720}$ ,  $\frac{240}{720}$ ,  $\frac{180}{720}$ ,  $\frac{144}{720}$ ,  $\frac{120}{720}$ , when added together, make  $\frac{1044}{720}$ , that is, 1  $\frac{9}{20}$ ; try it in money.

7thly,  $\frac{1}{2}$ ,  $\frac{2}{3}$ ,  $\frac{3}{4}$ ,  $\frac{4}{5}$ , and  $\frac{5}{6}$ , that is,  $\frac{360}{720}$ ,  $\frac{480}{720}$ ,  $\frac{540}{720}$ ,  $\frac{576}{720}$  and  $\frac{600}{720}$ , when added together, make  $\frac{2556}{720}$ , that is, 3  $\frac{11}{20}$ .

8thly, The sum of the mixt numbers 7  $\frac{1}{3}$  and 8  $\frac{1}{4}$  is 15  $\frac{7}{12}$ ; for the sum of the fractions is  $\frac{7}{12}$  by the second example, and the sum of the whole numbers is 15.

9thly, 5  $\frac{2}{3}$  added to 7  $\frac{4}{5}$  gives 13  $\frac{7}{15}$ ; for the sum of the fractions is  $\frac{7}{15}$ , by the fourth example; and the whole number 1, added to the whole numbers 5 and 7, gives 13.

10thly, 8  $\frac{1}{2}$ , 9  $\frac{2}{3}$ , 10  $\frac{3}{4}$ , 11  $\frac{4}{5}$ , 12  $\frac{5}{6}$ , added together, make 53  $\frac{11}{20}$ ; for the fractions themselves make 3  $\frac{11}{20}$  by the seventh example, and the whole number 3 added to the rest makes 53.

11thly,

1<sup>st</sup>ly, The whole number 2 added to the fraction  $\frac{3}{4}$  gives  $\frac{11}{4}$ ; for the whole number 2 may be considered as a fraction, whose denominator is unity; now  $\frac{2}{1}$  and  $\frac{3}{4}$ , when reduced to the same denomination, are  $\frac{8}{4}$  and  $\frac{3}{4}$ , which added together make  $\frac{11}{4}$ .

Thus also may unity be added to any fraction whatever, when subtraction requires it; but better thus: unity may be made a fraction of any denomination whatever, provided the numerator be equal to the denominator, by art. 2d: suppose then I would add unity to  $\frac{2}{3}$ ; I suppose unity equal to  $\frac{3}{3}$ , and this added to  $\frac{2}{3}$  makes  $\frac{5}{3}$ : again, unity added to  $\frac{3}{5}$  makes  $\frac{8}{5}$ , because  $\frac{5}{5}$  and  $\frac{3}{5}$  make  $\frac{8}{5}$ .

### *Of the Subtraction of Fractions.*

10. Whenever a less fraction is to be subtracted from a greater, they must be prepared as in addition; that is, they must be reduced to the same denomination, if they be not so already; then, *subtracting the numerator of the less fraction from that of the greater, put down the remainder with the common denominator under it.* In the case of mixt numbers, *subtract first the fraction of the lesser number from that of the greater, and then the lesser whole number from the greater; but if, as it often happens, the greater number has the lesser fraction belonging to it, then an unit must be borrowed from the whole number and added to the fraction, as intimated in the close of the last article.*

### *Examples of Subtraction in Fractions.*

1<sup>st</sup>,  $7\frac{3}{10}$  subtracted from  $1\frac{4}{10}$  leaves  $1\frac{1}{10}$ , just in the same manner as 3 shillings subtracted from 4 shillings leave 1 shilling.

2<sup>d</sup>ly,  $\frac{3}{4}$  subtracted from  $\frac{5}{6}$ , that is  $\frac{18}{24}$  subtracted from  $\frac{20}{24}$ , leaves  $\frac{2}{24}$ , or  $\frac{1}{12}$ . So  $\frac{3}{4}$  of a pound, or 15 shillings,

15 shillings subtracted from  $\frac{5}{6}$  of a pound, or 16 shillings and 8 pence, leaves  $1\frac{1}{2}$  of a pound, that is, 1 shilling and 8 pence.

3dly,  $7\frac{1}{3}$  subtracted from  $8\frac{1}{2}$ , that is,  $7\frac{2}{6}$  subtracted from  $8\frac{3}{6}$ , leaves  $1\frac{1}{6}$ .

4thly,  $7\frac{3}{4}$  subtracted from  $8\frac{1}{4}$ , that is,  $7\frac{3}{4}$  subtracted from  $7\frac{5}{4}$ , leaves  $\frac{2}{4}$ , or  $\frac{1}{2}$ ; for here the greater number having the less fraction belonging to it, I borrow an unit from the whole number 8, and so reduce it to 7; and then this unit, under the name of  $\frac{4}{4}$ , I add to the fraction  $\frac{1}{4}$ , and so make it  $\frac{5}{4}$ .

5thly,  $7\frac{2}{3}$  subtracted from  $8\frac{1}{2}$ , that is  $7\frac{4}{6}$  subtracted from  $8\frac{3}{6}$ , that is,  $7\frac{4}{6}$  subtracted from  $7\frac{9}{6}$ , leaves  $\frac{5}{6}$ .

6thly,  $7\frac{3}{5}$  subtracted from 8, that is,  $7\frac{3}{5}$  subtracted from  $7\frac{5}{5}$ , leaves  $\frac{2}{5}$ .

### *Of the Multiplication of Fractions.*

11. To multiply by a whole number is to take the multiplicand as often as that whole number expresses: therefore *to multiply by a mixt number is, not only to take the multiplicand as often as the integral part expresses, but also to take such a part or parts of it over and above, as is expressed by the fraction annexed.* Thus 10 multiplied by  $2\frac{1}{2}$  produces 25: for as  $2\frac{1}{2}$  is a middle number between 2 and 3, so the product ought to be a middle number between 20 and 30, that is, 25: In like manner 10 multiplied by  $1\frac{1}{2}$  produces 15, and being multiplied by  $\frac{1}{2}$  produces 5: therefore, to multiply by a proper fraction is nothing else but to take such a part or parts of the multiplicand as is expressed by that fraction. Certainly to take 10 twice and half of it over, once and half of it over, no times and half of it over, (which last is taking the half of 10), are operations of the same kind, and differ only in degree one from another; and therefore, if the two former operations pass by the name of multiplication, this last ought to do

do so too; and if there be any absurdity in the case, it lies in the name, and not in the thing.

Arithmetic was at first employed about whole numbers only, and thus far the name of multiplication was adequate enough, except in the case of unity. But it being afterwards considered, that no quantity whatever could be called an unit, that was not further divisible; and consequently, that there was not only an infinity of fractional numbers below unity, but also an infinity of mixt numbers between any two whole numbers whatever; it was judged, rightly enough, that the art of Arithmetic would not be perfect till its operations extended themselves to this sort of number also; and this being done without changing their names, it was then that the name of multiplication became too scanty for the thing signified: this therefore ought to be attributed to the unavoidable want of foresight in the first imposers, and not to any imperfection in the science itself. This is no more than the case of many other arts and sciences, that have outgrown their names. Thus Geometry, that originally and properly signified no more than the art of surveying, is now defined to be a science treating of the nature and properties of all figures, or rather of the different modifications of extension and space; so that now surveying is the least and lowest part of that science. Thus Hydrostatics, which originally signified no more than the art of weighing bodies in water, or rather the art of finding out the specific gravities of bodies by weighing them in water, is now made the name of a science, which treats of the nature and properties of fluids in general; and the several properties of air and mercury, so far as they are fluids, fall under the consideration of Hydrostatics, as properly as those of water.

But perhaps it may be further urged, that to take the half of any quantity, is not to multiply, but to divide it. To which I answer; that it is impossible  
to

to take the half of any quantity without dividing it by 2; and consequently, that to multiply by  $\frac{1}{2}$  has the same effect as to divide by 2; but this does not prove that multiplication is the same as division, but only that these two operations, how contrary soever, may be made to do each other's business, which is no mystery to any one who is the least conversant in Arithmetic; and will be further explained in the next article.

*A fraction may be multiplied by a whole number two ways; either by multiplying the numerator by that number, or else by dividing the denominator by the same, where such a division is possible: thus if the fraction  $\frac{5}{6}$  be to be multiplied by 2, the product will either be  $\frac{10}{6}$  by doubling the numerator, or  $\frac{5}{3}$  by halving the denominator: this is evident from the 6th art. because a fraction will be equally encreased, whether it be by encreasing the numerator, or by diminishing the denominator.*

*If a fraction be to be multiplied by a fraction, multiply the numerator and denominator of the multiplicand, by the numerator and denominator of the multiplier respectively, and the fraction thence arising will be the product sought: thus if it was required to multiply  $\frac{4}{5}$  by  $\frac{2}{3}$ , or (which amounts to the same thing) if it was required to determine how much is  $\frac{2}{3}$  of  $\frac{4}{5}$ , the answer would be  $\frac{8}{15}$ ; and the reason is plain; for  $\frac{1}{3}$  of  $\frac{4}{5}$  is  $\frac{4}{15}$ , by the sixth art. because making the denominator three times greater makes the fraction three times less: but if  $\frac{1}{3}$  of  $\frac{4}{5}$  be  $\frac{4}{15}$ , then  $\frac{2}{3}$  of  $\frac{4}{5}$  ought to be twice as much, that is  $\frac{8}{15}$ ; therefore to determine the amount of  $\frac{2}{3}$  of  $\frac{4}{5}$ , the numerator and denominator of  $\frac{4}{5}$  must be multiplied respectively by the numerator and denominator of  $\frac{2}{3}$ ; and the same reason will hold good in all other instances.*

*If a whole number is to be multiplied by a fraction, either change the multiplier and multiplicand one for another, and then proceed as above directed; or else*

else consider the multiplicand as a fraction whose denominator is unity, and so proceed according to the rule for multiplying one fraction by another; by which means both rules will be contracted into one. Thus 6, or  $\frac{6}{1}$ , multiplied into  $\frac{2}{3}$ , produces  $\frac{12}{3}$ , or 4.

If the multiplier, or multiplicand, or both, be mixt numbers, they must first be reduced to improper fractions by the third art. and then be multiplied according to the general rule.

*Examples of Multiplication in Fractions.*

1<sup>st</sup>,  $\frac{2}{3}$  of  $\frac{7}{8}$ , multiplying numerators together, and denominators together, is  $\frac{14}{24}$ , or  $\frac{7}{12}$ ; and so we find it in any particular case; for  $\frac{7}{8}$  of a pound are 17 shillings and 6 pence; and  $\frac{2}{3}$  of 17 shillings and 6 pence, that is (by the 5th art.)  $\frac{1}{3}$  of 35 shillings, is 11 shillings and 8 pence; therefore  $\frac{2}{3}$  of  $\frac{7}{8}$  of a pound are 11 shillings and 8 pence, which will also be found to be the value of  $\frac{7}{12}$  of a pound.

Here we may observe once for all, that whenever two fractions are to be multiplied together, the product will be the same, whichever it is that multiplies the other, just as it is in whole numbers, and for the same reason; for if  $\frac{7}{8}$  be to be multiplied by  $\frac{2}{3}$ , then the numbers 7 and 8 must be respectively multiplied by 2 and 3; but if  $\frac{2}{3}$  is to be multiplied by  $\frac{7}{8}$ , then the numbers 2 and 3 must be respectively multiplied by 7 and 8, which amounts to the same thing; whence it follows, that  $\frac{2}{3}$  of  $\frac{7}{8}$  come to the same as  $\frac{7}{8}$  of  $\frac{2}{3}$ : to confirm this, we have seen already that  $\frac{2}{3}$  of  $\frac{7}{8}$  of a pound amount to 11 shillings and 8 pence; let us in the next place enquire into the value of  $\frac{7}{8}$  of  $\frac{2}{3}$  of a pound: now  $\frac{2}{3}$  of a pound are 13 shillings and 4 pence; and  $\frac{7}{8}$  of 13 shillings and 4 pence, that is,  $\frac{1}{8}$  of 93 shillings and 4 pence, is 11 shillings and 8 pence; therefore  $\frac{2}{3}$  of  $\frac{7}{8}$  of a pound are the same as  $\frac{7}{8}$  of  $\frac{2}{3}$  of a pound, since both amount to 11 shillings and 8 pence.

2<sup>dly</sup>,

2dly,  $\frac{2}{3}$  of  $\frac{5}{8}$  of  $\frac{9}{10}$  are  $\frac{90}{180}$ , or  $\frac{1}{2}$ ; for  $2 \times 5 \times 9$  make 90, and  $3 \times 6 \times 10$  make 180: thus  $\frac{9}{10}$  of a pound are 18 shillings; and  $\frac{5}{8}$  of 18 shillings are 15 shillings; and  $\frac{2}{3}$  of 15 shillings are 10 shillings; which are  $\frac{1}{2}$  of a pound.

3dly,  $\frac{3}{4}$  of  $\frac{3}{4}$  of  $\frac{3}{4}$  are  $\frac{27}{64}$ : thus  $\frac{3}{4}$  of a pound are 15 shillings; and  $\frac{3}{4}$  of 15 shillings are 11 shillings and 3 pence; and  $\frac{3}{4}$  of 11 shillings and three pence are 8 shillings and 5 pence farthing; which will also be found to be the value of  $\frac{27}{64}$  of a pound.

4thly, The mixt number  $6 \frac{3}{4}$  multiplied by the whole number 7, or the whole number 7 multiplied by the mixt number  $6 \frac{3}{4}$ , will produce in either case  $47 \frac{1}{4}$ : for the mixt number  $6 \frac{3}{4}$  being reduced (by the 3d art.) to an improper fraction becomes  $\frac{27}{4}$ ; which being multiplied by 7, or  $\frac{7}{1}$ , makes  $\frac{189}{4}$ , or, when reduced to a mixt number,  $47 \frac{1}{4}$ .

This multiplication may also be made another way, thus:  $\frac{3}{4}$  multiplied by 7 makes  $\frac{21}{4}$ , that is, (by the 2d art.)  $5 \frac{1}{4}$ ; put down the fraction  $\frac{1}{4}$ , and keep the 5 in reserve; then 6 multiplied by 7 makes 42, which, with the 5 in reserve, makes 47; therefore the whole product is  $47 \frac{1}{4}$  as before.

5thly,  $3 \frac{3}{4}$  multiplied by  $2 \frac{2}{3}$ , that is,  $\frac{15}{4}$  multiplied by  $\frac{8}{3}$ , makes  $\frac{120}{12}$ , that is, 10: thus  $3 \frac{3}{4}$  of a pound are 3 pounds 15 shillings; and twice 3 pounds 15 shillings is 7 pounds 10 shillings; moreover,  $\frac{2}{3}$  of 3 pounds 15 shillings, or  $\frac{1}{3}$  of 7 pounds 10 shillings, is 2 pounds 10 shillings; and these 2 pounds 10 shillings, added to the former part of the product, to wit, 7 pounds ten shillings, give ten pounds for the whole product: therefore  $3 \frac{3}{4}$  of a pound multiplied by  $2 \frac{2}{3}$  make 10 pounds.

6thly,

6thly,  $96 \frac{1}{2}$  multiplied by  $24 \frac{1}{3}$ , that is,  $\frac{193}{2}$ , multiplied by  $\frac{73}{3}$ , gives  $\frac{24089}{6}$ , that is, (by the 2d art.)  $2348 \frac{1}{6}$ .

7thly,  $36 \frac{1}{4}$  multiplied into itself, that is,  $\frac{145}{4}$ , multiplied by  $\frac{145}{4}$ , makes  $\frac{21025}{16}$ , that is,  $1314 \frac{1}{16}$ .

Before I put an end to this article, I do not know whether it will be thought worth my while to take notice of a very absurd question sometimes bandied about, wherein it is required to multiply  $\frac{1}{3}$  of a pound by  $\frac{1}{2}$  of a pound: I call this a very absurd question, because there is no manner of propriety in it; for in the very idea and definition of multiplication, the multiplicator at least is supposed to be an abstract number, or fraction; otherwise, what can be the meaning of taking the multiplicand as often, or as much of it, as is expressed by the multiplicator? If by multiplying  $\frac{1}{3}$  of a pound by  $\frac{1}{2}$  of a pound, be meant no more than multiplying  $\frac{1}{3}$  of a pound by  $\frac{1}{2}$ , why is the word pound expressed in the multiplicator? and if there be any other meaning in it, why does not the proposer explain it, since it is not expressed in the question? Let him tell me what he means by multiplying 1 pound by 1 pound, and I will soon undertake to answer his question. But if he neither can nor will do this, the question neither deserves nor is capable of an answer. I am not ignorant of another question more frequently used than this, and of equal nonsense, if custom had not explained it; and that is, to multiply 3 yards by 2 yards, and the like; whereby is meant, I suppose, to assign the number of square yards contained in a rectangled parallelogram, or long square, 3 yards in length, and 2 yards in breadth; but if this be the sense put upon that question by common consent, that is all the title  
it

it has to it, there being no such thing either expressed, or so much as implied, in the terms of the question.

### A L E M M A.

12. *Let  $n$  be any whole number, mixt number, or fraction; I say then that the quotient of  $n$  divided by any fraction is equal to the product of  $n$  multiplied into the reverse of that fraction: as for instance,*

Let  $n$  be divided by  $\frac{3}{2}$ ; I say that the quotient of  $n$  divided by  $\frac{3}{2}$  will be equal to the product of  $n$  multiplied by  $\frac{2}{3}$ : for let  $q$  be the quotient of  $n$  divided by  $\frac{3}{2}$ ; that is, let  $q$  be a number expressing how often the fraction  $\frac{3}{2}$  is contained in  $n$ ; then will  $\frac{3}{2}$  multiplied by  $q$  be equal to  $n$ , from the nature of multiplication; but the product of  $\frac{3}{2}$  multiplied by  $q$  is the same with the product of  $q$  multiplied by  $\frac{3}{2}$ , that is,  $\frac{3}{2}$  of  $q$ , by the last article; therefore  $n$  is equal to  $\frac{3}{2}$  of  $q$ ; therefore  $\frac{1}{3}$  of  $n$  is equal to  $\frac{1}{2}$  of  $q$ ; therefore  $\frac{2}{3}$  of  $n$  are equal to  $q$ ; but  $\frac{2}{3}$  of  $n$  is the product of  $n$  multiplied by  $\frac{2}{3}$ ; therefore the product of  $n$  multiplied by  $\frac{2}{3}$  is equal to  $q$ ; but the quotient of  $n$  divided by  $\frac{3}{2}$  was  $q$ , by the supposition; therefore the quotient of  $n$  divided by  $\frac{3}{2}$ , is equal to the product of  $n$  multiplied by  $\frac{2}{3}$ . Q. E. D.

### C O R O L L A R Y.

Hence may the rule of division be at any time changed into that of multiplication, only by inverting the terms of the divisor, and then multiplying instead of dividing. The same will also obtain in whole numbers, if they be considered as fractions whose denominators are units: thus to divide  $n$  by 2, that is,  $\frac{2}{1}$ , will have the same effect as to multiply it by  $\frac{1}{2}$ , as was hinted in the foregoing article.

*Of the Division of Fractions.*

13. *The division of fractions, like all other division, is, to find how often one fraction, called the divisor, is contained in another, called the dividend; and that which shews this, is called the quotient, whether it be a whole number, a mixt number, or a proper fraction: for in fractional division the quotient is always intended to be exact, without any remainder, and therefore must sometimes be a whole number, sometimes a mixt number, and sometimes a proper fraction. Thus, if 18 is to be divided by 6, the quotient will be 3; because 18 contains 6 3 times: but if 21 is to be divided by 6, the quotient will be  $3\frac{1}{2}$ ; because 21 contains 6 three times, and half of it over and above: lastly, if 3 is to be divided by 6, the quotient will be  $\frac{1}{2}$ ; because here the divisor, being greater than the dividend, cannot be so much as once contained in it, and therefore the quotient in this case must be a proper fraction, that is,  $\frac{1}{2}$ , since 3 is just the half of 6.*

*A fraction may be divided by a whole number two ways; either by dividing the numerator by that whole number when possible, or else by multiplying the denominator by the same: thus the half of  $\frac{6}{7}$  may be taken, that is,  $\frac{6}{7}$  may be divided by 2, either by halving the numerator, and the quotient will be  $\frac{3}{7}$ , or else by doubling the denominator, and then the quotient will be  $\frac{6}{14}$ , both which amount to the same thing, by the 6th and 7th articles.*

*If the divisor be a fraction, the quotient may be had by multiplying the dividend into the inverted divisor, according to the rules of multiplication already laid down: thus if  $\frac{4}{5}$  is to be divided by  $\frac{2}{3}$ , the quotient will be the same as the product of  $\frac{4}{5}$  multiplied by  $\frac{3}{2}$ , that is,  $\frac{12}{10}$ , or  $1\frac{1}{5}$ ; the demonstration whereof is contained in the last article.*

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And

And here again, as well as in the eleventh article, we are to observe, that *if either divisor or dividend, or both, be mixt numbers, they must be reduced to improper fractions* before the general rule can have place; and that, *if either or both be whole numbers, they must be considered as fractions whose denominators are units.*

From the general rule of division before laid down it follows, that every fraction may be considered as the quotient of the numerator divided by the denominator, and that, whether the terms of the fraction under consideration be whole numbers, or (which sometimes happens) mixt numbers, or even pure fractions: a demonstration of this last case will serve for all, since mixt numbers may be reduced to fractions, and whole numbers may be considered as fractions whose denominators are units. Let the fraction proposed be  $\frac{\frac{4}{5}}{\frac{2}{3}}$

I say, that this fraction is equal to the quotient arising from the division of the numerator  $\frac{4}{5}$  by the denominator  $\frac{2}{3}$ : to demonstrate which, multiply both  $\frac{4}{5}$  the numerator, and  $\frac{2}{3}$  the denominator, by  $\frac{3}{2}$  the inverted denominator, and the fraction will be changed into this,  $\frac{\frac{4}{5} \cdot \frac{3}{2}}{\frac{2}{3} \cdot \frac{3}{2}}$ , or  $\frac{\frac{12}{10}}{\frac{6}{6}}$ , being of the same value with the former, by the 6th art. but the quotient of  $\frac{4}{5}$  divided by  $\frac{2}{3}$  is also  $\frac{12}{10}$  as above: therefore *the fraction  $\frac{\frac{4}{5}}{\frac{2}{3}}$  is equal to the quotient arising from the division of the numerator by the denominator*: and the same way of reasoning may be used in any other instance. *This consideration is of very great use in Algebra, where quantities are very often so generally expressed, that there is no other way of representing the quotient, but by a fraction whose numerator is the dividend, and denominator the divisor.* Hence also we are taught how to reduce a complicated fraction, into a simple one, whose numerator and denominator are whole numbers; to wit, by dividing the numerator by

by the denominator : thus we see that  $\frac{4}{\frac{5}{2}}$  is the same as  $\frac{1}{\frac{5}{8}}$ .

*Other Examples of Division in Fractions.*

$1\frac{1}{2}$ ,  $\frac{5}{6}$  divided by  $\frac{3}{4}$ , or, which is the same thing,  $\frac{5}{6}$  multiplied into  $\frac{4}{3}$ , makes  $\frac{20}{18}$ , or  $1\frac{2}{9}$ ; which shews that  $\frac{3}{4}$  is contained once, and  $\frac{1}{9}$  part of it over and above, in  $\frac{5}{6}$ : for a further confirmation of this,  $\frac{5}{6}$  of a pound are 16 shillings and 8 pence; and  $\frac{3}{4}$  of a pound are 15 shillings: now 15 shillings are once contained in 16 shillings and 8 pence, and there is 1 shilling and 8 pence over; which 1 shilling and 8 pence is just  $\frac{1}{9}$  of 15 shillings. To prevent oversights, the learner is to remember, that it is *the terms of the divisor only that are to be inverted*, and not those of the dividend: thus to divide  $\frac{5}{6}$  by  $\frac{3}{4}$  is the same as to multiply  $\frac{5}{6}$  into  $\frac{4}{3}$ , but not the same as to multiply  $\frac{6}{5}$  into  $\frac{4}{3}$ .

2dly,  $\frac{9}{10}$  divided by  $\frac{1}{3}$ , or multiplied into  $\frac{3}{1}$ , make  $\frac{27}{10}$ , or  $2\frac{7}{10}$ , which may be confirmed like the former: for  $\frac{9}{10}$  of a pound are 18 shillings; and  $\frac{1}{3}$  of a pound is 6 shillings and 8 pence: now 6 shillings and 8 pence are twice contained in 18 shillings, and there are 4 shillings and 8 pence over; which 4 shillings and 8 pence will be found by the 5th art. to be just  $\frac{7}{10}$  of 6 shillings and 8 pence.

3dly, The whole number 10 divided by  $2\frac{2}{3}$ , that is  $\frac{10}{1}$  divided by  $\frac{8}{3}$ , or multiplied into  $\frac{3}{8}$ , makes  $\frac{30}{8}$ , or  $3\frac{3}{4}$ .

4thly,  $2\frac{2}{3}$  divided by  $\frac{1}{10}$ , or  $\frac{8}{3}$  divided by  $\frac{1}{10}$ , or multiplied into  $\frac{10}{1}$ , makes  $\frac{80}{3}$ , or  $26\frac{2}{3}$ .

5thly,  $16\frac{1}{3}$  divided by  $1\frac{1}{6}$ , that is,  $\frac{49}{3}$  divided by  $\frac{7}{6}$ , or multiplied into  $\frac{6}{7}$ , makes  $\frac{294}{7}$ , or 42.

*Further Observations concerning Multiplication and Division in Fractions.*

14. *When two fractions are multiplied together, or one is divided by the other, it often happens, that though the original fractions be both in their least terms, yet the product, or quotient from them, shall be otherwise, and require a further reduction: as for instance, the fractions  $\frac{5}{6}$  and  $\frac{2}{15}$  are both in their least terms; and yet, if they be multiplied together, their product  $\frac{45}{90}$  is so far from being in its least terms, that it may be reduced to  $\frac{1}{2}$ : so again in division,  $\frac{2}{15}$  and  $\frac{1}{15}$  are fractions both in their least terms; and yet if the latter be divided by the former, the quotient  $\frac{15}{30}$  is reducible to  $\frac{1}{2}$ . It may not be amiss, therefore, to enquire into the cause of this, and see whether the original fractions may be so prepared beforehand, as that the product, or quotient, shall always come out in its least terms. First then, as to the multiplication of  $\frac{5}{6}$  and  $\frac{2}{15}$ ; here it is easy to see, that the product of  $\frac{5}{6}$  and  $\frac{2}{15}$  multiplied together, will just amount to the same as that of  $\frac{5}{15}$  into  $\frac{2}{6}$ , the denominators of the fractions being interchanged; this, I say, is certain from the operation itself; for the same numbers are multiplied together in both cases; but these last fractions are far from being in their least terms, the former,  $\frac{5}{15}$  being reducible to  $\frac{1}{3}$ , and the latter  $\frac{2}{6}$  to  $\frac{1}{3}$ ; but after these new fractions  $\frac{1}{3}$  and  $\frac{1}{3}$  are reduced to their least terms  $\frac{1}{3}$  and  $\frac{1}{3}$ ; their product  $\frac{1}{9}$  will be the same in value with that of the original fractions, and at the same time will be in its least terms. Thus then we see that, to have the product in its least terms, care must be taken, not only to reduce the original fractions as low as possible, but after that, to interchange their denominators, and then again to reduce these*

*new fractions to their least terms, and lastly, to multiply these reduced fractions one into another.*

*The same manner of practice will also serve for division, after it is reduced to the rule of multiplication: as for example; the quotient of  $\frac{1}{1} \frac{5}{6}$  divided by  $\frac{1}{1} \frac{9}{6}$ , is the same with the product of  $\frac{1}{1} \frac{5}{6}$  multiplied into  $\frac{1}{1} \frac{6}{9}$ ; and this again is the same with the product of  $\frac{1}{1} \frac{5}{6}$  multiplied into  $\frac{1}{1} \frac{6}{6}$ , as above; but because the fractions  $\frac{1}{1} \frac{5}{6}$  and  $\frac{1}{1} \frac{6}{6}$  are not in their lowest terms, they must be reduced to  $\frac{5}{6}$  and  $\frac{6}{6}$  before it can be expected that their product  $\frac{2}{3} \frac{5}{4}$  should be in its least terms. Thus we have reduced the two compendiums of multiplication and division, not only to one rule instead of two, as they are commonly given out, but also to such a rule as carries its own evidence along with it.*

*N. B. What was here done by interchanging the denominators, and keeping the numerators in their places, may as well be done by interchanging the numerators, and keeping the denominators in their places, the reason of both being the same.*

### *Of the Rule of Proportion in Fractions.*

15. The rule of proportion in fractions is so much the same with the rule of proportion in whole numbers, that nothing more needs to be said of it, except to illustrate it by an example or two.

#### *Examples of the Rule of Proportion in Fractions.*

1<sup>st</sup>, If  $\frac{1}{4}$  give  $\frac{1}{5}$ , what will  $\frac{1}{6}$  give? Here  $\frac{1}{5}$  and  $\frac{1}{6}$  multiplied together give  $\frac{1}{30}$ ; and this divided by  $\frac{1}{4}$ , (or multiplied by  $\frac{4}{1}$ ) quotes  $\frac{4}{30}$ , or  $\frac{2}{15}$ , which is an answer to the question.

2<sup>dly</sup>, If 2  $\frac{2}{3}$  give 3  $\frac{3}{4}$ , what will 4  $\frac{4}{5}$  give? These mixt numbers, being by the 3<sup>d</sup> art. reduced to improper fractions, will stand thus: If  $\frac{8}{3}$  give  $\frac{15}{4}$ , what will  $\frac{24}{5}$  give? Here  $\frac{15}{4}$  and  $\frac{24}{5}$  multiplied together

give  $\frac{180}{10}$  or 18; and this divided by  $\frac{3}{2}$ , quotes  $6\frac{2}{3}$ , which is an answer to the question.

3dly, If  $\frac{1}{2}$  a yard cost  $\frac{1}{3}$  of a pound, what will  $\frac{1}{4}$  of an ell cost? Here it must be observed, that an ell is  $\frac{5}{4}$  of a yard, and consequently that  $\frac{1}{4}$  of an ell is  $\frac{1}{4}$  of  $\frac{5}{4}$  or  $\frac{5}{16}$  of a yard; so that the question may be stated thus: If  $\frac{1}{2}$  of a yard cost  $\frac{1}{3}$  of a pound, what will  $\frac{5}{16}$  of a yard cost? Here  $\frac{1}{3}$  and  $\frac{5}{16}$  multiplied together make  $\frac{5}{48}$ , and this divided by  $\frac{1}{2}$  quotes  $\frac{5}{24}$  of a pound, or 4 shillings and 2 pence; which therefore is an answer to the question.

### *The Reduction of Proportion from fractional to integral Terms.*

Whenever two fractions are proposed, as  $\frac{2}{3}$  and  $\frac{4}{5}$ , whose proportion is desired in whole numbers, reduce the fractions first to the same denomination by the 8th art. that is, in the present case, to  $\frac{10}{15}$  and  $\frac{12}{15}$ ; then you will have  $\frac{2}{3}$  to  $\frac{4}{5}$  as  $\frac{10}{15}$  is to  $\frac{12}{15}$ ; but  $\frac{10}{15}$  is to  $\frac{12}{15}$  as 10 to 12, or as 5 to 6; therefore  $\frac{2}{3}$  is to  $\frac{4}{5}$  as 5 to 6: here we may observe, that *though the finding of the common denominator be necessary for understanding the reason of the rule, yet it is not at all necessary for the practice of it*; for to what purpose is it to find the common denominator, to throw it away again when we have done? In practice, therefore, multiply the numerator of the fraction which is the first in the proportion, by the denominator of the second, and then the numerator of the second fraction by the denominator of the first, and the two products will exhibit respectively the proportion of the first fraction to the second in whole numbers, as was evident in the foregoing example.

### *Of the Extraction of Roots in Fractions.*

16. As every fraction is squared, or multiplied into itself, by squaring both the numerator and denominator.

nator (see art. 11.), so, *è converso*, the square root of every fraction will be obtained by extracting the square root both of the numerator and denominator: thus the square of  $\frac{3}{4}$  is  $\frac{9}{16}$ , and the square root of  $\frac{9}{16}$  is  $\frac{3}{4}$ . But here care must be taken, whenever the square root of a fraction is to be extracted, that the fraction itself be first reduced to its simplest terms, by the 7th art. otherwise the fraction may admit of a square root, and yet this root may not be discovered: thus, if it was required to extract the square root of the fraction  $\frac{18}{32}$ , it would be impossible to obtain the root either of 18 or 32; and yet when this fraction is reduced to its least terms  $\frac{9}{16}$ , its square root will be found to be  $\frac{3}{4}$ .

When the square root of a number cannot be extracted exactly, it is usual to make an approximation by the help of decimals, or otherwise, and so to approach as near to the value of the true root as occasion requires. Now in the case of a fraction, if the square root of neither the numerator nor denominator can be exactly obtained, there will be no necessity however for two approximations, because such a fraction may be easily reduced to another of the same value, whose denominator is a known square: as for instance; suppose the square root of  $46\frac{1}{5}$ , or  $\frac{231}{5}$  was required: I multiply both the numerator and denominator of this fraction by 5, and so reduce it to  $\frac{1155}{25}$ : Here the denominator 25 is a known square number, whose root is 5; and the square root of 1155 is 34 nearly; therefore, the square root of the fraction proposed is nearly  $\frac{34}{5}$ , or  $6\frac{4}{5}$ . But, after all, the best way of extracting the square root of a vulgar fraction, is by throwing it into a decimal fraction, as will be shewn hereafter.

Note, That whatever has here been said concerning the extraction of the square root of fractions may

easily be applied, mutatis mutandis, to the extraction of the cube root, &c.

## Of Decimal Fractions.

### And first of their Notation.

17. A decimal fraction is a fraction whose denominator is 10, or 100, or 1000, or 10000, &c. and this denominator is never expressed, but always understood by the place of the figure it belongs to: for as all figures on the left hand of the place of units rise in their value, according to their distances from it, in a decuple proportion; so all figures on the right hand of the place of units sink in their value in a subdecuple proportion; as for instance; the number 345.6789, where 5 stands in the place of units, is to be read thus; *three hundred forty-five, six tenths, seven hundredth parts, eight thousandth parts, nine ten thousandth parts*: or the decimal parts may be read thus; *six thousand seven hundred eighty nine ten thousandth parts*; the denominator being ten thousand, because the last figure 9, according to the former way of reckoning, stands in the place of ten-thousandth parts. The reason of this latter way of reading is plain; for  $\frac{6}{10}$  are  $\frac{6000}{10000}$ , and  $\frac{7}{100}$  are  $\frac{700}{10000}$ , and  $\frac{8}{1000}$  are  $\frac{80}{10000}$ , and  $\frac{6000}{10000}$ ,  $\frac{700}{10000}$ ,  $\frac{80}{10000}$ , and  $\frac{9}{10000}$ , all added together, make  $\frac{6789}{10000}$ .

Cyphers are used in the expression of decimals as well as whole numbers, and for the same reason, Thus .067 may be read either *no tenths, six hundredth parts, seven thousandth parts*; or *sixty seven thousandth parts*. But cyphers on the right hand of a decimal number (if nothing follows them) are as insignificant as cyphers on the left hand of a whole number; and yet cyphers are sometimes placed after decimals, for  
the

*the sake of regularity, or when we want to increase the number of decimal places.*

From what has here been said, it will be easy to multiply or divide any number by 10, 100, 1000, &c. only by removing the separating point towards the right or left hand. Thus the number 345·6789 being multiplied by 10, becomes 3456·789; and being multiplied by 100, becomes 34567·89: and the same number 345·6789 being divided by 10, becomes 34·56789; and being divided by 100, becomes 3·456789: thus again, the number 345 being divided by 10000, becomes ·0345; for to divide by 10000, is the same thing as to remove the separating point 4 degrees towards the left hand, if there be any separating point in the number given; but if there be none, as in the present case, then to put a separating point four degrees towards the left hand, which in this example cannot be done, but by the help of a cypher in the first decimal place.

*Of the Addition and Subtraction of Decimal Fractions.*

18. *The chief advantage of decimal arithmetic above that of common fractions, consists in this, that in decimals all operations are performed as in whole numbers: this will presently appear from the several parts of decimal arithmetic, as they come now to be treated of in order; and first of addition and subtraction.*

*Addition and subtraction in decimals are performed after the same manner as in whole numbers, care being taken, that like parts be placed under one another: as for example, ·567 are added to ·89 thus;*

·89	·89	·890.
·567 subtracted thus;	·567; or thus;	·567.
1·457:	·323;	·323.

### *Of the Multiplication of Decimal Fractions.*

19. *Multiplication of decimals is also performed as in whole numbers, no regard being had to the decimals as such, till the product is obtained; but then, so many decimal places must be cut off from the right hand of the product, as are contained both in the multiplicator and multiplicand: as for instance; let it be required to multiply 4.56 by 2.3: here, considering both factors as whole numbers, I multiply 456 by 23, and find the product to be 10488; but then, considering that there was one decimal in the multiplicator, and two in the multiplicand, I cut off three decimal places from the right hand of the product, and the true product stands thus; 10.488.*

To shew the reason of this operation, let the two factors be reduced to simple fractions according to the common way, and we shall have 2.3 equal to  $\frac{23}{10}$ , and 4.56 equal to  $\frac{456}{100}$ , and these two fractions multiplied together make  $\frac{10488}{1000}$ ; divide by 1000, which is done by cutting off the three last figures, according to art. the 17th, and the quotient will be 10.488. Another example may be this: let it be required to multiply 45600 by 2.3: the product of 45600 multiplied by 23 is 1048800: but as there were two decimals in the given multiplicator, and none in the multiplicand; I cut off two decimal places from the last product, and the true product will be found to be 10488.00, or 10488. Lastly, let it be required to multiply .000456 by .23: here, neglecting the initial cyphers in the multiplicand, I multiply 456 by 23, and the product is 10488: then I consider, that there are two decimal places in the multiplicator, and six in the multiplicand, and consequently that eight decimal places are to be cut off from the last product; but the last product

consists

consists of only 5 places; therefore I place three cyphers to the left hand, with the separating point before them, and so make the true product  $\cdot 00010488$ .

There are *various compendiums of this sort of multiplication* to be met with in *Oughtred and others* \*; but they are such as, by a little exercise, any one tolerably well grounded in this part of Arithmetic will easily discover of himself as they lie in his way.

### *Of the Division of Decimal Fractions.*

20. *Division in decimal fractions is performed, first by considering them as whole numbers, and dividing accordingly; and then cutting off from the right hand of the quotient, as many decimal places as the dividend hath more than the divisor.* The reason whereof is manifest from the 19th article: for since the divisor and quotient multiplied together are to make the dividend, the divisor and quotient ought to have as many decimal places between them, as there are in the dividend; therefore the quotient alone ought to have as many decimal places as the dividend hath more than the divisor.

Example the 1<sup>st</sup>; Let it be proposed to divide  $10\cdot 488$  by  $2\cdot 3$ : here dividing the whole number  $10488$  by the whole number  $23$ , I find the quotient to be  $456$ : but then considering that there were 3 decimal places in the dividend, and but one in the divisor, I cut off two places from the right hand of the quotient, and so make the true quotient  $4\cdot 56$ .

Example 2<sup>d</sup>; Let it be proposed to divide  $5678\cdot 9$  by  $\cdot 06$ : here, because there are two decimal places in the divisor, and but one in the dividend, I supply the deficient place by putting a cypher after the dividend, thus,  $5678\cdot 90$ ; then dividing the whole number  $567890$  by the whole number,  $6$  (for since  $6$  is now considered as a whole number, the cypher

\* See Dr. Hutton's Practical Arithmetic, p. 80; or see Wingate's Arithmetic, section, or article, 396.

before it may be neglected), I find the quotient to be 94648, which is not to be sunk, because the dividend was made to have as many decimal places as the divisor; but as this quotient is not exact, if for a greater degree of exactness I would continue it to any number of decimal places, suppose 2, instead of one cypher after the dividend, I would have put three, and then the quotient would have come out 94648.33, and this quotient is much more exact than the former, as lying between 94648.33 and 94648.34: but it ought further to be observed concerning this quotient, that if the division was to be continued *in infinitum*, the figures in the decimal places would be all 3's: this is evident from the work; for the two last dividends are the same, and therefore they must all be the same.

*To reduce a Vulgar Fraction to a Decimal Fraction.*

21. Since every fraction may be considered as the quotient of the numerator divided by the denominator (see art. 13th), we have an easy rule for reducing a vulgar fraction to a decimal fraction, which is as follows; *put as many cyphers after the numerator as are equal in number to the number of decimal places whereof you intend your reduced fraction to consist, and call these cyphers decimal; and then dividing the numerator by the denominator, the quotient will be a decimal number equal to the fraction first proposed, or perhaps a mixt number, if the fraction proposed was an improper one.*

Example 1<sup>st</sup>; Let this fraction  $\frac{3}{49}$  be proposed to be reduced to a decimal one consisting of four decimal places; here putting 4 decimal cyphers after the numerator 3, I divide 3.0000 by 49, and the quotient uncorrected is 612: but now considering that there were 4 decimal places in the dividend, and none in the divisor, and consequently that four decimal places are to be cut off from the quotient, whereas it consists but of three; I supply this defect of places by a cypher at the left hand, and so make the quotient .0612.

Example

Example 2d; Let this fraction  $\frac{7}{16}$  be proposed to be reduced to a decimal fraction, consisting, if possible, of six places: here dividing 7·000000 by 16, I find the true quotient to be ·4375, the two last cyphers in the dividend being useless.

*Note.* When this division runs *ad infinitum*, it will be impossible for the reduction to be exact in a finite number of terms; but an approximation may be made, that shall come nearer to the quotient than the least assignable difference, by taking more and more terms.

*To reduce the decimal Parts of any Integer to such other Parts as that Integer is usually divided into.*

22. To explain this rule, and to give an example of it at the same time; let ·345 of a pound sterling, that is, three hundred forty-five thousandth parts of a pound, be given to be reduced into shillings, pence, and farthings: here then I observe, that as any number of pounds, multiplied by 20, will give as many shillings as are equal to the pounds, so any decimal parts of a pound, multiplied by 20, will give as many shillings, and decimal parts of a shilling, as are equivalent to the decimal parts of a pound; and so on as to pence and farthings: multiplying therefore ·345 by 20, the product is 6 and ·900, or 6·9, which signifies, that ·345 of a pound are equivalent to six shillings and nine tenths of a shilling, which is usually written thus; 6·9 shillings: again, multiplying this last decimal ·9 by 12 for pence, I find that ·9 of a shilling are equivalent to 10·8 pence: lastly, multiplying ·8 by 4 for farthings, I find that ·8 of a penny are equivalent to 3·2 farthings; as for the ·2 of a farthing, I neglect it, there being no lower denomination, or at least not intending to descend any lower; and so I find ·345 of a pound to amount to six shillings and ten pence three farthings.

*To reduce the common Parts of any Integer into equivalent decimal Parts of the same.*

23. This reduction being the reverse of the former, it might be performed by division, as that was by multiplication; but, when all things are considered, I do not know whether the following method may not be thought as easy and as intelligible as any: let it then be required to reduce 2 hours 34 minutes 56 seconds into equivalent decimal parts of a day. Now in one day there are 86400 seconds; and in 2 hours 34 minutes 56 seconds there are 9296 seconds; therefore 2 hours 34 minutes 56 seconds are equivalent to  $\frac{9296}{86400}$  of one day: reduce this vulgar fraction to an equivalent decimal, by the last article but one, and you will find it to be  $\cdot 10759$ ; therefore 2 hours 34 minutes 56 seconds are equivalent to  $\cdot 10759$  of one day. But there is one article still remains to be adjusted, and that is, to how many decimal places the foregoing fraction must be reduced, so as to express accurately enough the parts of a day to a second of time. Now to know this, I consider that one second of time is  $\frac{1}{86400}$  of one day; therefore I reduce  $\frac{1}{86400}$  to a decimal fraction, at least as far as to the first significant figure, and find it to be  $\cdot 00001$ ; whence I conclude, that to express the parts of a day to a second of time by any decimal, that decimal must not consist of fewer than 5 places, because there were 5 places in the decimal fraction  $\cdot 00001$ . Now to shew that the decimal fraction above found, to wit,  $\cdot 10759$  expresses the time proposed to a second, reduce it back again, by the last art. and you will find it amount to 2 hours 34 minutes 55 8 seconds.

For another example, let us take the reverse of that in the last art. that is, let it be proposed to reduce

duce 6 shillings 10 pence 3 $\frac{1}{2}$  farthings into equivalent decimal parts of a pound: one pound contains 960 farthings, or 9600 tenths of a farthing; and 6 shillings 10 pence 3 $\frac{1}{2}$  farthings contain 3312 tenths of a farthing; therefore 6 shillings 10 pence 3 $\frac{1}{2}$  farthings are equivalent to  $\frac{3312}{9600}$  of a pound; but  $\frac{1}{9600}$  being reduced to a decimal, is, .0001, &c. wherein the first significant figure is in the 4th place; therefore I reduce the fraction  $\frac{3312}{9600}$ , to four decimal places, and they amount to .3450, that is, .345 of a pound; so that in this particular case three decimal places are sufficient to express exactly the sum proposed.

*Of the Extraction of the Square Root in Decimal Fractions.*

24. Having treated of the multiplication and division of decimal fractions, it would be altogether needless to say any thing concerning the rule of proportion, which is but a particular application of both: therefore I shall now pass on to the extraction of the square root, at least so far as it concerns decimal fractions. *There are but few square numbers, or such as will admit of an exact square root, in comparison of the rest; and therefore, whenever a number is proposed to have its square root extracted, the artist must first determine with himself, to how many decimal places it is proper the root should be continued; and then, by annexing decimal cyphers, if need be, to the right hand of the number proposed, he must make twice as many decimal places there as the root is to consist of; after this, he must put a point over the place of units, and then, passing by every other figure, he must point in like manner all the rest, both to the right hand, and to the left: by this*

means

means the number will be prepared, and *the square root* may be *extracted as in whole numbers*, provided that *so many decimal places be cut off from the root* when obtained, *as were first designed*.

Example 1*st*; Let the root of 2345·6 be required to two decimal places. The number, when prepared, stands thus, 2345·6000, or as a whole number, thus, 23456000; and its square root, when extracted, will be 4843 nearly; and therefore 48·43 will be the root sought. To try this root 48·43, multiply it into itself, and the four first figures of the square will be 2345, which are all true; nor can it be expected any more should be so, because there were but four places true in the root, no notice being taken of the rest; but had the root been extracted true to 5 places, that is, to as many places as the original square consisted of, it would then have been 48·431; multiply this number into itself, and 5 of the first figures of the product, taken with the least error, will be 2345·6, which is the original square itself.

Example 2*d*; Let the root of ·0023456 be required to 5 decimal places. Here putting a cypher in the place of units to direct the punctuation thus, 0·0023456000, I extract the square root of 23456000 as of a whole number, and find it to be 4843, as above: but, considering that this root is to be sunk 5 places, I put a cypher to the left hand, and so make the true root ·04843.

That the supposed square ought to have twice as many decimal places as the root, is evident, both *à priori*, and *à posteriori*: *à priori*, because in extracting the square root, two figures are brought down from the square for every single figure gained in the root; and *à posteriori*, because the root multiplied into itself is to produce the square; and therefore, from the nature of multiplication, the square ought to have twice as many decimal places as the root.

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THE  
ELEMENTS OF ALGEBRA.

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B O O K I.

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*The Definition of Algebra.*

*Art. 1.* **I** SHALL not here detain the young student with a long historical account of the rise and progress of Algebra; nor even so much as with either the etymology or signification of the word; which would contribute but very little to his information, till he has made a further progress in the science itself, and whereof he will find enough in *Dr. Wallis*, and others. Nor indeed is it a subject altogether so proper at this time to be insisted upon; this art, like many others, having now considerably outgrown its name, and being often employed in arithmetical operations very different from what its name imports. All I shall advance then, by way of definition, is, that *Algebra*, in the modern sense of the word, is the art of computing by symbols, that is, generally speaking, by letters of the alphabet; which, for the simplicity and distinctness both of their sounds and characters, are much more commodious for this purpose than any other symbols or marks whatever.

In this way of notation, it is usual to substitute letters not only for such quantities as are unknown, and consequently such as cannot well be represented otherwise, but also for known quantities themselves, in order to keep them distinct one from another, and to form general conclusions. As for instance; suppose it was demanded of me, what two numbers are those, whose sum is 48, and whose difference is 14: here, if I only put  $x$ , or some other letter, for one of the unknown quantities, and use the known ones 48 and 14 as I find them in the problem, I shall only come to this particular conclusion, to wit, that the greater number is 31, and the lesser 17, which numbers will answer both the conditions of the problem. But if, instead of the known numbers 48 and 14, I substitute the general quantities  $a$  and  $b$  respectively, and so propose the problem thus; *What two numbers are those, whose sum is  $a$ , and whose difference is  $b$ ?* I shall then come to this general conclusion, viz. that *Half the sum of  $a$  and  $b$  will be the greater number, and half their difference will be the less*: which general theorem will suit not only the particular case abovementioned, but also all other cases of this problem that can possibly be proposed. How I come by these two conclusions, will be sufficiently shewn in the course of this work; as also many other advantages attending this way of substituting letters for known quantities, besides those already mentioned.

What I have here said, was only to illustrate in some measure the definition already given of Algebra, and to shew, that letters are there used, not so much to signify particular quantities as such, as to signify the relation they have to one another in any problem or computation. From all which it may be observed, that letters represent quantities in Algebra just in the same manner as they do persons in common life, when two or more persons are distinctly to be considered, with regard to any compact, law-suit, or in any other relation whatever.

N. B. *A single quantity is sometimes represented by two or more letters, when it is considered as the product of the quantities signified by those letters singly: thus  $ab$  is the product of the multiplication of  $a$  and  $b$ ; and  $abc$  is the product arising from the continual multiplication of  $a$ ,  $b$ , and  $c$ . But of this more particularly under the head of Multiplication.*

*Of affirmative and negative Quantities in Algebra.*

2. Algebraic quantities are of two sorts, affirmative and negative: *an affirmative quantity is a quantity greater than nothing, and is known by this sign  $+$ : a negative quantity is a quantity less than nothing, and is known by this sign  $-$ : thus  $+a$  signifies that the quantity  $a$  is affirmative, and is to be read thus, plus  $a$ , or more  $a$ :  $-b$  signifies that the quantity  $b$  is negative, and must be read thus, minus  $b$ , or less  $b$ .*

The possibility of any quantity's being less than nothing is to some a very great paradox, if not a downright absurdity; and truly so it would be, if we should suppose it possible for a body or substance to be less than nothing. But quantities, whereby the different degrees of qualities are estimated, may be easily conceived to pass from affirmation through nothing into negation. Thus a person in his fortunes may be said to be worth 2000 pounds, or 1000, or nothing, or  $-1000$ , or  $-2000$ ; in which two last cases he is said to be 1000 or 2000 pounds worse than nothing: thus a body may be said to have two degrees of heat, or one degree, or no degree, or  $-$  one degree, or  $-$  two degrees: thus a body may be said to have two degrees of motion downwards, or one degree, or no degree, or  $-$  one degree, or  $-$  two degrees, &c. Certain it is, that all contrary quantities do necessarily admit of an intermediate state, which alike partakes of both extremes, and is best represented by a cypher or 0: and if it is proper to

say, that the degrees on either side this common limit are greater than nothing; I do not see why it should not be as proper to say of the other side, that the degrees are less than nothing; at least in comparison to the former. That which most perplexes narrow minds, in this way of thinking, is, that *in common life most quantities lose their names when they cease to be affirmative, and acquire new ones so soon as they begin to be negative*: thus we call negative goods, debts; negative gain, loss; negative heat, cold; negative descent, ascent, &c.: and in this sense, indeed, it may not be so easy to conceive how a quantity can be less than nothing, that is, how a quantity under any particular denomination can be said to be less than nothing, so long as it retains that denomination. But the question is, whether of two contrary quantities under two different names, one quantity under one name may not be said to be less than nothing, when compared with the other quantity, though under a different name? whether any degree of cold may not be said to be further from any degree of heat, than is lukewarmth, or no heat at all? Difficulties that arise from the imposition of scanty and limited names, upon quantities which in themselves are actually unlimited, ought to be charged upon those names, and not upon the things themselves, as I have formerly observed upon another occasion; see Introduction, art. 11. *In Algebra*, where quantities are abstractedly considered, without any regard to degrees of magnitude, the names of quantities are as extensive as the quantities themselves; so that *all quantities that differ only in degree one from another, how contrary soever they may be one to another, pass under the same name*; and affirmative and negative quantities are only distinguished by their signs, as was observed before, and not by their names; the same letter representing both: these signs therefore in Algebra carry the same distinction along with them as do particles and adjectives sometimes in com-

mon language, as in the words convenient and inconvenient, happy and unhappy, good health and bad health, &c.

These affirmative and negative quantities, as they are contrary to one another in their own natures, so likewise are they in their effects; a consideration which, if duly attended to, would remove all difficulties concerning the signs of quantities arising from addition, subtraction, multiplication, division, &c.: for the result of working by affirmative quantities in all these operations is known; and, therefore, like operations in negative quantities may be known by the rule of contraries.

Before we proceed any further, it may not be amiss to advertise, that *if a quantity has no sign before it, it must always be taken to be affirmative*; and that *if it has no numeral coefficient before it, unity must always be understood*: thus  $2a$  signifies  $+2a$ , and  $a$  signifies  $1a$  or  $+1a$ .

By the *numeral coefficient* of a quantity, I mean, the number or fraction by which that quantity is multiplied: thus  $2a$  signifies twice  $a$ , or  $a$  taken twice, and the coefficient is 2:  $\frac{3}{4}a$ , or  $\frac{3a}{4}$  signifies  $\frac{3}{4}$  of the quantity  $a$ , and the coefficient is  $\frac{3}{4}$ .

N. B. *The sign of a negative quantity is never omitted; nor the sign of an affirmative one, except when such an affirmative quantity is considered by itself, or happens to be the first in a series of quantities succeeding one another*: thus we do not often mention the quantity  $+a$ , but the quantity  $a$ ; nor the series  $+a - b - c + d$ , but the series  $a - b - c + d$ . We shall now consider the several operations of algebraic quantities.

### *Of the Addition of algebraic Quantities.*

3. This article I shall divide into several paragraphs: as,

1st, *Whenever two or more quantities of the same denomination, and which have the same sign before them, are to be added together, put down the sum of their numeral coefficients with the common sign before it, and the common denominator after it: thus*  $+2a$  and  $+3a$  added together make  $+5a$ , for the same reason as 2 dozen and 3 dozen added together make 5 dozen: thus again,  $-3ab$ ,  $-4ab$ , and  $-5ab$ , when added together, make  $-12ab$ ; for the same reason as several debts added together make a greater debt.

2d, *If two quantities of the same denomination which have different signs before them are to be added together, put down only the difference of their numeral coefficients, with the common denominator after it, and the sign of the greater quantity before it: for in this case, the quantities to be added being contrary one to another, the less quantity, on which side soever it lies, will always destroy so much of the other as is equal to itself. Thus*  $+5a$  added to  $-2a$  makes  $+3a$ ; as if a person owes me 5000 pounds upon one account, to whom I owe 2000 upon another, the balance upon the whole will be 3000 pounds on my side. If it be objected, that this is subtraction, and not addition; I answer, that the addition of  $-2a$  will at any time have the same effect as the subtraction of  $+2a$ : but I deny that the addition of  $-2a$  is the same, or will have the same effect as the subtraction of  $-2a$ . Other examples of this case may be these;  $+7a$  added to  $-7a$  gives 0;  $+3a$  added to  $-12a$  gives  $-9a$ :  $+a$  added to  $-5a$  gives  $-4a$ ;  $+5a$  added to  $-a$  gives  $+4a$ ;  $+\frac{1}{3}a$  added to  $-\frac{1}{4}a$  gives  $+\frac{1}{12}a$ , &c.

3d, *When many quantities of the same denomination are to be added together, whereof some are affirmative and some negative, reduce them first to two, by adding all the affirmative quantities together, and all the negative ones, and then to one by the last paragraph. Thus*  $+10a - 9a + 8a - 7a$ , when added together,

together, make  $2a$ ; for  $+10a$  and  $+8a$  make  $+18a$ ;  $-9a$  and  $-7a$  make  $-16a$ ; and  $+18a$  and  $-16a$  make  $+2a$ .

4th, Quantities of different denominations will not incorporate, and therefore cannot otherwise be added together, than by placing them in any order one after another, with their proper signs before them, except the first, whose sign, if affirmative, may be omitted. Thus  $+2a$  and  $-3b$  and  $+4c$  and  $-5d$ , when added together, make  $2a - 3b + 4c - 5d$ : thus  $a$  and  $b$  added together make  $a + b$ ; and hence it is, that whenever two quantities are found with this sign  $+$  betwixt them, it signifies the sum arising from the addition of those two quantities together: thus if  $a$  stands for 7, and  $b$  stands for 3,  $a + b$  will stand for 10, and so of the rest: but if  $-b$  is to be added to  $a$ , the sum must be written down thus,  $a - b$ ; for to add  $-b$ , is the same as to subtract  $+b$ .

5th, Compound quantities, whose members are all of different denominations, are likewise incapable of being added any other way than by being placed one after another without altering their signs: thus  $3a + 4b$  added to  $5c - 6d$  can only make  $3a + 4b + 5c - 6d$ . But if the members are not all of different denominations, it may then be convenient to place one compound quantity under another, with like parts under like, as far as it can be done, as in the following examples;

$$\begin{array}{rcl}
 a + b & \dagger & \text{For } a \text{ and } a \text{ added together make} \\
 a - b & & 2a; \text{ and } +b \text{ and } -b \text{ added together} \\
 \hline
 2a & * \dagger & \text{destroy one another, and so make 0} \\
 & & \text{or } *; \text{ which character in Algebra is} \\
 & & \text{always used to signify a vacant place.}
 \end{array}$$

$$\begin{array}{rcl}
 2x - 3a + 4b - & 5c + 6d - 7e & * \\
 10x + 9a - 8b - & 7c - 6d & * - 5f \\
 \hline
 12x + 6a - 4b - 12c & * - 7e - 5f. &
 \end{array}$$

*Note*, That in the addition, subtraction, and multiplication of compound algebraic quantities, *it matters little which way the work is carried on, whether from right or left, or from left to right*, because here are no reserves made for higher places.

*Of the Subtraction of algebraic Quantities.*

4. *Whenever a single algebraic quantity is to be subtracted from another quantity, whether simple or compound, first change the sign of the quantity to be subtracted, that is, if it be affirmative, make it, or at least call it, negative, and vice versa, and then add it so changed to the other*: for since (as was before hinted) the subtracting of any one quantity from another, is the same in effect as adding the contrary; and since changing the sign of the quantity to be subtracted, renders that quantity just contrary to what it was before, it is evident, that after such a change it may be added to the other, and that the result of this addition will be the same with that of the intended subtraction. Thus may the rule of subtraction, by changing the sign of the quantity to be subtracted, be at any time changed into that of addition, just as the rule of division in fractions, by inverting the terms of the divisor, was changed into that of multiplication. As for example,  $+b$  subtracted from  $a$  leaves  $a - b$ , because  $-b$  added to  $a$  makes  $a - b$ ; so that  $a - b$  may be considered either as the sum of  $a$  and  $-b$  added together, or as the remainder of  $+b$  subtracted from  $a$ , or as the difference between  $a$  and  $b$ , or as the excess of  $a$  above  $b$ , all which amount to the same thing: as if  $a$  signifies 7, and  $b$  3,  $a - b$  must stand for 4, and so of the rest.

The rule of subtraction here given is universal, though there will not be always occasion to have recourse to it: for suppose  $3a$  is to be subtracted from  $7a$ , every one's common sense will inform him, that

there must remain  $4a$ , just as threescore subtracted from seven score leaves four score.

*Other Examples of algebraic Subtraction may be these that follow.*

1<sup>st</sup>,  $7a$  subtracted from  $5a$  leaves  $-2a$ , because  $-7a$  added to  $+5a$  makes  $-2a$ , by the 2<sup>d</sup> paragraph of the last article.

2<sup>d</sup>,  $9a$  subtracted from 0 leaves  $-9a$ , because  $-9a$  added to 0 makes  $-9a$ .

3<sup>d</sup>,  $12a$  subtracted from  $-3a$  leaves  $-15a$ , because  $-12a$  added to  $-3a$  makes  $-15a$ , by the first paragraph of the last article.

4<sup>th</sup>,  $-3a$  subtracted from  $-8a$  leaves  $-5a$ , because  $+3a$  added to  $-8a$  makes  $-5a$ .

5<sup>th</sup>,  $-7a$  subtracted from  $-3a$  leaves  $+4a$ , because  $+7a$  added to  $-3a$  makes  $+4a$ .

6<sup>th</sup>,  $-6a$  subtracted from 0 leaves  $+6a$ , because  $+6a$  added to 0 makes  $+6a$ .

7<sup>th</sup>,  $-5a$  subtracted from  $+5a$  leaves  $+10a$ , because  $+5a$  added to  $+5a$  makes  $+10a$ .

8<sup>th</sup>,  $-b$  subtracted from  $a$  leaves  $a+b$ , because  $+b$  added to  $a$  makes  $a+b$ , by the 4<sup>th</sup> paragraph of the last article.

9<sup>th</sup>,  $-2$  subtracted from 7 leaves 9, because  $+2$  added to 7 makes 9.

From the first of these examples it appears, that a greater quantity may be taken out of a less, but then the remainder will be negative; just as a gamester that has but 5 guineas about him may lose 7, but then there will remain a debt of 2 guineas upon him. By the last example it appears, that  $-2$  subtracted from 7 leaves 9, that is, that if a negative quantity be subtracted from an affirmative one, the affirmative quantity will be so far from being diminished thereby, that it will be increased; a principle which I fear will be found somewhat hard of digestion, especially by weak constitutions: therefore,

therefore, to strengthen my patient as far as lies in my power, I shall suggest to him the following considerations :

*1st*, In any subtraction, if the remainder and the less number added together make the greater, the subtraction is just : but in our case, the remainder 9 added to the less number  $-2$  makes the greater number 7 ; therefore  $-2$  subtracted from 7 leaves 9.

*2dly*, In all subtraction whatever, the remainder is the difference betwixt the greater number and the less ; but the difference between  $+7$  and  $-2$  is 9 ; therefore  $-2$  subtracted from  $+7$  leaves 9.

*3dly*, 7 is equal to  $9-2$  by the second paragraph of the last article ; therefore  $-2$  subtracted from 7 will have the same remainder as  $-2$  subtracted from  $9-2$  : but  $-2$  subtracted from  $9-2$  leaves 9 ; therefore  $-2$  subtracted from 7 leaves 9. In short, the taking away a defect, in any case whatever, will amount to the same as adding something real : as if an estate be incumbered with a mortgage or a rent-charge upon it, whoever takes off the incumbrance just so much encreases the value of the estate.

*4thly*, The less there is taken from 7, the more will be left : if nothing be taken, there will remain 7 ; therefore if less than nothing be taken, there ought to remain more than 7.

*5thly*, If, after all that has been said, or perhaps all that can be said in this abstracted way, some scruples still remain, let us apply the principle we have already advanced, and try whether we shall meet with any better success that way. Let it then be required to subtract the compound quantity  $a-2$  from the compound quantity  $6a+7$  : in order to this, I place  $a$  under  $6a$ , and  $-2$  under 7, and then subtract as follows ;  $a$  from  $6a$  and there remains  $5a$ ,  $-2$  from 7 and (if our assertion be true) there remains 9 ; therefore the whole remainder is  $5a+9$ . Now I dare appeal to every one's common sense,

sense, whether this subtraction be not just : for certain it is, that if  $a$  be subtracted from  $6a + 7$ , the remainder will be  $5a + 7$  ; and if so, then it is as certain, that if  $a - 2$  be subtracted, which is less than the former by 2, the remainder will be greater by 2, that is,  $5a + 9$ . But to proceed :

*Other Examples of the Subtraction of compound algebraic quantities may be these :*

$a + b$     † Thus  $7 - 3$ , or 4, subtracted from 7    \* + 12  
 $a - b$     + 3, or 10, leaves twice 3, or 6.     $3a + 7$

$$\begin{array}{r}
 * + 2b^{\dagger} \\
 \text{From} \quad 12x + 6a - 4b - 12c \quad * - 7e - 5f \\
 \text{Take} \quad 2x - 3a + 4b - 5c + 6d - 7e - * \\
 \hline
 \text{Remains} \quad 10x + 9a - 8b - 7c - 6d \quad * - 5f \\
 \hline
 \text{Proof} \quad 12x + 6a - 4b - 12c \quad * - 7e - 5f
 \end{array}$$

If never a member of the subtrahend be found to be of the same denomination with any member of the number from whence the subtraction is to be made, change the sign of every member of the subtrahend, and then add it to the other. As if  $5c - 6d$  is to be subtracted from  $3a - 4b$ , first change the sign of  $5c - 6d$ , and make it  $-5c + 6d$ , and then add it to the other, and you will have  $3a - 4b - 5c + 6d$  for the remainder.

### *Of the Multiplication of algebraic Quantities.*

*And first, how to find the sign of the product in multiplication, from those of the multiplicator and multiplicand given.*

5. Before we can proceed to the multiplication of algebraic quantities, we are to take notice, that  
 if

*if the signs of the multiplicator and multiplicand be both alike, that is, both affirmative, or both negative, the product will be affirmative; otherwise it will be negative: thus +4 multiplied into +3, or -4 into -3, produces in either case +12: but -4 multiplied into +3, or +4 into -3, produces in either case -12.*

If the reader expects a demonstration of this rule, he must first be advertised of two things: *first*, that numbers are said to be in arithmetical progression when they increase or decrease with equal differences, as 0, 2, 4, 6; or 6, 4, 2, 0; also as 3, 0, -3; 4, 0, -4; 12, 0, -12; or -12, 0, +12: whence it follows, that three terms are the fewest that can form an arithmetical progression; and that of these, if the two first terms be known, the third will easily be had: thus, if the two first terms be 4 and 2, the next will be 0: if the two first be 12 and 0, the next will be -12; if the two first be -12 and 0, the next will be +12, &c.

*2dly*, If a set of numbers in arithmetical progression, as 3, 2, and 1, be successively multiplied into one common multiplicator, as 4, or if a single number, as 4, be successively multiplied into a set of numbers in arithmetical progression, as 3, 2, and 1, the products 12, 8, and 4, in either case, will be in arithmetical progression.

This being allowed (which is in a manner self-evident), the rule to be demonstrated resolves itself into four cases:

*1st*, That +4 multiplied into +3 produces +12.

*2dly*, That -4 multiplied into +3 produces -12.

*3dly*, That +4 multiplied into -3 produces -12.

And *lastly*, that -4 multiplied into -3 produces +12. These cases are generally expressed in short thus: first + into + gives +; secondly - into + gives

gives  $-$  ; thirdly  $+$  into  $-$  gives  $-$  ; fourthly  $-$  into  $-$  gives  $+$ .

*Case 1st*, That  $+4$  multiplied into  $+3$  produces  $+12$ , is self-evident, and needs no demonstration; or, if it wanted one, it might receive it from the first paragraph of the third article; for to multiply  $+4$  by  $+3$  is the same thing as to add  $4 + 4 + 4$  into one sum; but  $4 + 4 + 4$  added into one sum gives  $+12$ , therefore  $+4$  multiplied into  $+3$  gives  $+12$ .

*Case 2d*. And from the second paragraph of the 3d art. it might in like manner be demonstrated, that  $-4$  multiplied into  $+3$  produces  $-12$ : but I shall here demonstrate in another way, thus: multiply the terms of this arithmetical progression  $4, 0, -4$ , into  $+3$ , and the products will be in arithmetical progression, as above; but the two first products are  $12$  and  $0$ ; therefore the third will be  $-12$ ; therefore  $-4$  multiplied into  $+3$  produces  $-12$ .

*Case 3d*. To prove that  $+4$  multiplied into  $-3$  produces  $-12$ ; multiply  $+4$  into  $+3, 0$ , and  $-3$  successively, and the products will be in arithmetical progression; but the two first products are  $12$  and  $0$ , therefore the third will be  $-12$ ; therefore  $+4$  multiplied into  $-3$  produces  $-12$ .

*Case 4th*. Lastly, to demonstrate, that  $-4$  multiplied into  $-3$  produces  $+12$ , multiply  $-4$  into  $3, 0$ , and  $-3$  successively, and the products will be in arithmetical progression; but the two first products are  $-12$  and  $0$ , by the second case; therefore the third product will be  $+12$ ; therefore  $-4$  multiplied into  $-3$  produces  $+12$ .

$$\begin{array}{rcl} \text{Cas. 2d, } +4, & 0, & -4 \\ & +3, & +3, & +3 \end{array} \qquad \begin{array}{rcl} \text{Cas. 3d, } +4, & +4, & +4 \\ & +3, & 0, & -3 \end{array}$$

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$$+12, \quad 0, \quad -12$$

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$$+12, \quad 0, \quad -12$$

$$\begin{array}{rcl} \text{Cas. 4th, } -4, & -4, & -4 \\ & +3, & 0, & -3 \end{array}$$

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$$-12, \quad 0, \quad +12$$

These

*X & Y the two first products*

These 4 cases may be also more briefly demonstrated thus :  $+4$  multiplied into  $+3$  produces  $+12$  ; therefore  $-4$  into  $+3$ , or  $+4$  into  $-3$  ought to produce something contrary to  $+12$ , that is,  $-12$  : but if  $-4$  multiplied into  $+3$ , produces  $-12$ , then  $-4$  multiplied into  $-3$  ought to produce something contrary to  $-12$ , that is,  $+12$  ; so that this last case, so very formidable to young beginners, appears at last to amount to no more than a common principle in Grammar, to wit, that two negatives make an affirmative ; which is undoubtedly true in Grammar, though perhaps it may not always be observed in languages.

*Of the Multiplication of simple Algebraic Quantities.*

6. These things premised, *the multiplication of simple algebraic quantities is performed, first by multiplying the numeral co-efficients together, and then putting down, after the product, all the letters in both factors, the sign (when occasion requires) being prefixed as above directed.* Thus  $4b$  multiplied into  $3a$  produces  $12ab$ .

Though this kind of language (for it is no more), like all others, be purely arbitrary, yet that a more rational one could not have been invented for this purpose, will appear by the following consideration. If any quantity, as  $b$ , is to be multiplied by any number, as 2, 3, or 4, the product cannot be better represented than by  $2b$ ,  $3b$ ,  $4b$ , &c. ; therefore if  $b$  is to be multiplied by  $a$ , the product ought to be called  $ab$  : but if  $b$  multiplied into  $a$  produces  $ab$ , then  $4b$  multiplied into  $a$  ought to produce 4 times as much, that is,  $4ab$  : lastly, if  $4b$  multiplied into  $a$  produces  $4ab$ , then  $4b$  multiplied into  $3a$  ought to produce 3 times as much, that is,  $12ab$ .

Hence it is, that *whenever in Algebra two or more letters are found together, as they stand in a word, without any thing between them, they signify the product*

duct arising from a continual multiplication of the quantities represented by them: thus  $ab$  signifies the product of  $a$  and  $b$  multiplied together; and  $abc$  signifies the product of the quantity  $ab$  multiplied into  $c$ : thus  $aa$  signifies the product of  $a$  multiplied into itself, or the square of  $a$ , and not  $2a$ ; and therefore whoever shews himself unable to distinguish betwixt  $2a$  and  $aa$ , discovers as great a weakness as one that is not able to distinguish betwixt 2 dozen and a dozen dozen or 12 times 12.

*It is a matter of no great consequence in what order the letters are placed in a product; for  $ab$  and  $ba$  differ no more from one another than 3 times 4, and 4 times 3: and yet it is convenient that a method be observed, lest like quantities be sometimes taken for unlike: therefore the best way will be, to give those letters the precedency in a product, that have it in the alphabet; except when an unknown quantity is multiplied by some known one, and then it is usual to place the known quantity before it.*

*Note.* For the signification of this mark  $\times$ , see introductory. at the close of the 7th article. Note also, that this mark  $=$  is a mark of equality, shewing that the quantities between which it stands are equal to each other, and must be read as the sense requires: thus  $2 \times 6 = 3 \times 4 = 12$  may be read thus;  $2 \times 6$  equal  $3 \times 4$  equal to 12: or thus;  $2 \times 6$  is equal to  $3 \times 4$ , which is equal to 12.

### *Examples of Simple Algebraic Multiplication.*

$$\begin{array}{ll}
 1^{\text{st}}, 4ab \times 5a = 20aab. & 2^{\text{d}}, -5ab \times 6bc = -30abc. \\
 3^{\text{d}}, 6ac \times -7bd = -42abcd. & 4^{\text{th}}, -7a \times -b = +7ab. \\
 5^{\text{th}}, x \times 3x = 3xx. & 6^{\text{th}}, -x \times -x = +xx. \\
 7^{\text{th}}, -5ab \times +3 = -15ab. & 8^{\text{th}}, \frac{2}{3}a \times \frac{4}{5}b = \frac{8}{15}ab.
 \end{array}$$

*Distinctions to be observed betwixt Addition and Multiplication.*

That the young algebraist may not confound the operations of addition and multiplication, as is frequently done; I shall here set down some marks of distinction, which he ought to attend to.

As *first*,  $a$  added to  $a$  makes  $2a$ , but  $a$  multiplied into  $a$  makes  $aa$ .

*2dly*,  $a$  added to  $o$  makes  $a$ , but  $a$  multiplied into  $o$  makes  $o$ .

*3dly*,  $a$  added to  $-a$  makes  $o$ , but  $a$  multiplied into  $-a$  makes  $-aa$ .

*4thly*,  $-a$  added to  $-a$  makes  $-2a$ , but  $-a$  multiplied into  $-a$  makes  $+aa$ .

*5thly*,  $a$  added to  $1$  makes  $a+1$ , but  $a$  multiplied into  $1$  makes  $a$ .

*6thly*,  $2a$  added to  $-3b$  makes  $2a-3b$ , but  $2a$  multiplied into  $-3b$  makes  $-6ab$ .

For a further confirmation of the learner, I have added, by way of exercise in his algebraic language, the following equations; which I desire he would compute after me. Suppose  $a=7$ , and  $b=3$ : then we shall have *1st*,  $a+b=10$ . *2dly*,  $a-b=4$ . *3dly*,  $4a+5b=43$ . *4thly*,  $4a-5b=13$ . *5thly*,  $aa=49$ . *6thly*,  $ab=21$ . *7thly*,  $bb=9$ . *8thly*,  $aaa=343$ . *9thly*,  $aab=147$ . *10thly*,  $abb=63$ . *11thly*,  $bbb=27$ . *12thly*,  $aa+2ab+bb=49+42+9=100$ . *13thly*,  $aa-2ab+bb=49-42+9=16$ . *14thly*,  $aaa+3aab+3abb+bbb=343+441+189+27=1000$ . *15thly*,  $aaa-3aab+3abb-bbb=343-441+189-27=64$ .

*Of Powers and their Indexes.*

7. Whenever in multiplication a letter is to be repeated oftener than once, it is usual, by way of compendium, to write down the letter with a small figure after

after it, shewing how often that letter is to be repeated: thus instead of  $xx$  we write  $x^2$ , instead of  $xxx$  we write  $x^3$ , instead of  $xxxx$  we write  $x^4$ , &c. These products are called powers of  $x$ ; the figures representing the number of repetitions are called the indexes of those powers; and the quantity  $x$ , from whence all these powers arise, is called the root of these powers, or the first power of  $x$ ;  $x^2$  is called the second power of  $x$ ,  $x^3$  the third power,  $x^4$  the fourth power, &c. Vieta, Oughtred, and some other analysts, instead of small letters, used capitals; and instead of numeral indexes, distinguished these powers by names: thus Vieta in particular called  $x^2$ , *X square*;  $x^3$ , *X cube*;  $x^4$ , *X square-square*;  $x^5$ , *X square-cube*;  $x^6$ , *X cube-cube*;  $x^7$ , *X square-square-cube*, &c.: which names Oughtred contracted, and wrote them thus; *Xq*, *Xc*, *Xqq*, *Xqc*, *Xcc*, *Xqqc*, &c.: but now these names are pretty much out of use, except the two first, when applied to a line squared or cubed.

If we suppose  $x = 5$ , we shall have  $2x = 10$ ,  $x^2 = 25$ ,  $3x = 15$ ,  $x^3 = 125$ ,  $4x = 20$ ,  $x^4 = 625$ , &c.

The multiplication of these powers is easy: thus  $x^2 \times x^3 = x^5$ , because  $xx \times xxx = xxxxx$ : whence it may be observed, that the addition of indexes will always answer to the multiplication of powers, provided they be powers of the same quantity; for as  $2 + 3 = 5$ , so  $x^2 \times x^3 = x^5$ , &c.: but if they be powers of different quantities, their indexes must not be added: thus  $a^2 \times x^3 = a^2x^3$ , and  $a^2x^3 \times a^4x^5 = a^6x^8$ . And here it must be observed, that if a number be found between two letters, it must always be referred to the former letter; thus  $a^2x^3$  does not signify  $a \times 2x^3$ , but  $a^2 \times x^3$ .

### *The Multiplication of Surds.*

8. This mark  $\sqrt{\phantom{x}}$  signifies the square root of the number to which it is prefixed, and is generally prefixed to numbers whose square root cannot be otherwise expressed, either by whole numbers or fractions:

thus  $\sqrt{2}$  signifies the square root of 2 ;  $\sqrt{a}$  the square root of  $a$ , &c. *These roots are commonly called surd roots, or irrational roots, because their proportion to unity cannot be expressed in numbers.*

*Whenever two surd numbers are to be multiplied together, the shortest way will be, to multiply the numbers themselves one into the other without any regard to the radical sign, and then to prefix the radical sign to the product.* Thus if  $\sqrt{a}$  is to be multiplied into  $\sqrt{b}$ , the product will be  $\sqrt{ab}$ ; which I thus demonstrate: let  $\sqrt{a}=x$ , and  $\sqrt{b}=y$ ; then will  $x^2=a$ , and  $y^2=b$ , and  $x^2y^2=ab$ , and  $xy=\sqrt{ab}$ ; but  $xy$ , or  $x \times y = \sqrt{a} \times \sqrt{b}$  by the supposition; therefore,  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ . Thus  $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ .

These multiplications are of considerable use, not only in matters of speculation, but also in practice: for suppose I had occasion to multiply the square root of 2 into the square root of 3, if I had not this rule, I must first extract the root of 2, to what degree of exactness I think proper for my purpose: then again I must extract the root of 3 to the same degree of exactness; and lastly I must multiply these two roots together, before I can obtain the number wanted: but after it is known that  $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ , the whole operation will then be reduced to the extraction of the root of 6 only: nay, it sometimes happens, that two roots, though both irrational, shall have a rational product: thus  $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$ , and  $\sqrt{ab^2} \times \sqrt{ac^2} = \sqrt{a^2b^2c^2} = abc$ .

### *Of the Multiplication of Compound Algebraic Quantities.*

9. *The multiplication of compound algebraic quantities is performed, first by multiplying the multiplicand into every particular member of the multiplier, and then reducing the whole product into the least compass possible.*

As

As for example; let it be required to multiply this compound quantity  $6x - 7a - 8b$  into this compound quantity  $2x - 3a + 4b$ : here having put down the multiplicand, and the multiplier under it, and beginning at the left hand (for it is all one which way the operation is carried on), I multiply the whole multiplicand into  $2x$ , the first member of my multiplier, and the product is  $12xx - 14ax - 16bx$ , which I put down: then I multiply the multiplicand into  $-3a$ , the next member of the multiplier, and the product is  $-18ax + 21aa + 24ab$ ; whereof the first member  $-18ax$ , I place under  $-14ax$  before found, being of the same denomination, for the conveniency of adding; the rest, to wit,  $+21aa + 24ab$ , I place in the first line: this done, I now multiply by  $4b$ , the last member of the multiplier, and the product is  $24bx - 28ab - 32bb$ ; whereof I place  $24bx$  under  $-16bx$ , and  $-28ab$  under  $+24ab$ , and the last member  $-32bb$  I place in the first line, as having no quantity of the same denomination to join with it: lastly I reduce the whole product into the least compass possible; and it stands thus:  $12xx - 32ax + 8bx + 21aa - 4ab - 32bb$ . See the work:

$$\begin{array}{r}
 6x \quad -7a \quad -8b \\
 2x \quad -3a \quad +4b \\
 \hline
 12xx - 14ax - 16bx + 21aa + 24ab - 32bb \\
 \quad -18ax + 24bx \quad \quad -28ab \\
 \hline
 \text{Sum } 12xx - 32ax + 8bx + 21aa - 4ab - 32bb
 \end{array}$$

Example 2d.

$$\begin{array}{r}
 3x \quad +4a \quad -5b \\
 3x \quad -4a \quad +5b \\
 \hline
 9xx + 12ax - 15bx - 16aa + 20ab - 25bb \\
 \quad -12ax + 15bx \quad \quad +20ab \\
 \hline
 9xx \quad * \quad * \quad -16aa + 40ab - 25bb
 \end{array}$$

Example 3d.

$$6xx - 7ax + 8aa$$

$$2xx - 3ax + 4aa$$

---


$$12x^4 - 14ax^3 + 16a^2x^2 - 24a^3x + 32a^4$$

$$- 18ax^3 + 21a^2x^2 - 28a^3x$$

$$+ 24a^2x^2$$


---

$$12x^4 - 32ax^3 + 61a^2x^2 - 52a^3x + 32a^4$$

Example 4th.      Example 5th.      Example 6th.

$$a + b$$

$$a - b$$

$$a + b$$

$$a + b$$

$$a - b$$

$$a - b$$

---


$$aa + ab - bb$$

$$- ab$$


---

---


$$aa + ab + bb$$

$$+ ab$$


---

---


$$aa - ab + bb$$

$$- ab$$


---

$$aa \quad * \quad -bb$$

$$aa + 2ab + bb$$

$$aa - 2ab + bb$$

N. B. *A dash over two or more quantities signifies that all those quantities are to be taken into one conception, or to be considered as making up but one compound quantity: thus  $a + b \times c - d$  does not signify that which arises from multiplying  $b \times c$ , and then adding  $a - d$  to the product, as it might be mistaken without the dash; but it signifies the product of the whole quantity  $a + b$  multiplied into the whole quantity  $c - d$ .*

*The Proof of Compound Multiplication.*

10. In the 3d example we multiplied  $6xx - 7ax + 8aa$  into  $2xx - 3ax + 4aa$ , and the product amounted to  $12x^4 - 32ax^3 + 61a^2x^2 - 52a^3x + 32a^4$ : let us try this in numbers, and see how it will answer. In order to which, we may suppose  $a$  and  $x$  equal to any two numbers whatever; but the simplest way of trial will be to make

make  $a$  equal 1, and  $x=1$ ; and then we shall have in the multiplicand  $6xx=6$ ,  $-7ax=-7$ , and  $+8aa=+8$ , and  $6-7+8=7$ : therefore the multiplicand is 7: again, in the multiplier we have  $2xx=2$ ,  $-3ax=-3$ ,  $+4aa=+4$ , and  $2-3+4=3$ ; therefore the multiplier is 3: and 7 the multiplicand, multiplied into 3 the multiplier, gives 21 for the product. Let us now examine the several parts of the product as they are here represented in letters, and see whether they will amount to that number:  $12x^4=12$ ,  $-32ax^3=-32$ ,  $+61a^2x^2=+61$ ,  $-52a^3x=-52$ ,  $+32a^4=+32$ ; and  $12-32+61-52+32$  amount to just 21. This may serve as a proof to the work, though not a necessary one: for it is not impossible but there may be a consistency this way, and yet the work be false; but this will rarely happen, unless it be designed. But the work may still be confirmed by making  $a=1$ , and  $x=-1$ ; for then the multiplicand will be  $6+7+8=21$ ; and the multiplier  $2+3+4=9$ ; and the product  $12+32+61+52+32=189$ , which is the same with the product of 21 the multiplicand multiplied into 9 the multiplier.

*How general Theorems may be obtained by  
Multiplication in Algebra.*

11. From these algebraic multiplications are derived and demonstrated many very useful theorems in all the parts of Mathematicks; whereof I shall just give the learner a taste, and then proceed to another subject.

In the fourth example of compound multiplication we found, that  $a+b$  multiplied into  $a-b$  produced  $aa-bb$ ; whence I infer, that *The sum and difference of any two numbers multiplied together will give the difference of their squares, and vice versa*: for  $a$  and  $b$  will represent any two numbers at pleasure;  $a+b$  their sum,  $a-b$  their difference, and  $aa-bb$  the difference of their squares: thus, if we assume any two numbers whatever, suppose 7 and 3, the difference of their

squares is  $49 - 9$ , or  $40$ ; and  $10$  their sum, multiplied into  $4$  their difference, makes also  $40$ .

But here I am to give notice once for all, that instances in numbers serve well enough to illustrate a general theorem, but they must not by any means be looked upon as a proof of it; because a proposition may be true in some particular cases instanced in, and yet fail in others; but whenever a proposition is found to be true *in speciebus*, that is, in letters or symbols, it is a sufficient demonstration of it, because these are universal representations.

In the 5th example it was shewn, that  $a + b$  multiplied into itself produced  $aa + 2ab + bb$ ; whence I infer, that *If a number be resolved into any two parts whatever, the square of the whole will be equal to the square of each part, and the double rectangle, or product of the multiplication of those parts, added together*: thus, if the number  $10$  be resolved into  $7$  and  $3$ ,  $100$  the square of  $10$  the whole, will be equal to  $49$  the square of  $7$ , and  $9$  the square of  $3$ , and  $42$  the double product of  $7$  and  $3$  multiplied together: for  $49 + 9 + 42 = 100$ .

In the 6th example we found, that  $a - b$  multiplied into itself, produced,  $aa - 2ab + bb$ ; whence I infer, that *If from the sum of the squares of any two numbers be subtracted the double product of those numbers, there will remain the square of their difference*: for  $aa + bb$  is the sum of the squares of  $a$  and  $b$ , and  $2ab$  is their double product, and  $aa - 2ab + bb$  was found to be the square of  $a - b$ , that is, the square of the difference of  $a$  and  $b$ : thus, in the numbers  $7$  and  $3$ , the square of  $7$  is  $49$ , the square of  $3$  is nine, and the sum of their squares is  $58$ ; and if from this be subtracted the double product  $42$ , the remainder will be  $16$ , the square of  $4$ , that is, the square of the difference of the numbers  $7$  and  $3$ .

These two last theorems are in substance the fourth and seventh propositions of the second book of *Euclid*.

*Of the Division of simple Algebraic Quantities.*

13. *The division of simple algebraic quantities, where it is possible in integral terms, is performed, first, by dividing the numeral coefficient of the dividend by the numeral coefficient of the divisor, and then putting down after the quotient all the letters in the dividend that are not in the divisor; the sign of the quotient in division being determined by those of the divisor and dividend, just in the same manner as the sign of the product in multiplication is determined by those of the multiplier and multiplicand; that is, if the sign of the divisor and dividend be both alike, whether they be both affirmative, or both negative, the quotient will be affirmative, otherwise it will be negative: thus if the quantity  $-12ab$  is divided by  $-3a$ , the quotient will be  $+4b$ ; which I thus demonstrate: In all division whatever, the quotient ought to be such a quantity as, being multiplied by the divisor, will make the dividend; therefore, to enquire for the quotient in our case, is nothing else but to enquire what number or quantity multiplied into  $-3a$ , the divisor, will produce  $-12ab$ , the dividend. First then I ask, what sign multiplied into  $-$ , the sign of the divisor, will give  $-$ , the sign of the dividend? and the answer is  $+$ ; therefore  $+$  is the sign of the quotient: in the next place I enquire, what number multiplied into 3, the coefficient of the divisor, will give 12, the coefficient of the dividend? and the answer is 4; therefore 4 is the coefficient of the quotient: lastly I enquire, what letter multiplied into  $a$ , the letter of the divisor, will produce  $ab$ , the denominator, or literal part of the dividend? and the answer is  $b$ ; therefore  $b$  is the letter of the quotient: and thus at last we have the whole quotient, which is  $+4b$ . And this way of reasoning will carry the learner through all the other cases.*

*Examples of simple Division in Algebra.*

Example 1st,  $4ab \mid 24abbc \ (6bc.$

2d,  $+7 \mid -35ab \ (-5ab.$

3d,  $-x \mid -3xx \ (\div 3x.$

4th,  $-9ab \mid +72ab \ (-8.$

5th,  $-4a^3 \mid -60a^8 \ (\div 15a^5.$

6th,  $4x^2 \mid 60x^9 \ (\div 15x^7.$

7th,  $\div 4a^3x^2 \mid -60a^8x^9 \ (-15a^5x^7.$

8th,  $b \mid \frac{2}{3}ab \ (\frac{2}{3}a.$

9th,  $\frac{2}{3} \mid \frac{4}{5}b \ (\frac{6}{5}b.$

*Of the Notation of Algebraic Fractions.*

Whenever a division according to the foregoing method is found impossible, the quotient cannot be otherwise expressed than by a fraction, whose numerator is the dividend, and denominator the divisor; see the Introduction, art. 13. As, if it was required to divide  $a$  by  $b$ , which division is impossible according to the foregoing rule, the quotient must be expressed by this fraction  $\frac{a}{b}$ , which is usually read

thus,  $a$  by  $b$ , that is,  $a$  divided by  $b$ , or the quotient of  $a$  divided by  $b$ : for in Algebra the word *by* is, generally speaking, appropriated to division, as the word *into* is to multiplication.

If the numerator, or denominator, or both, be compound quantities, the respective fractions must be written thus:

$$\frac{a \div b}{c}, \quad \frac{a}{b \div c}, \quad \frac{a \div b}{c \div d}.$$

If a division be partly possible according to the foregoing rules, and partly impossible, it must be pursued as far as it is possible, and the rest must be represented by a fraction, as in common division: thus if  $ad \div bd \div c$  was to be divided by  $d$ , the quotient would be  $a \div b \div \frac{c}{d}$ .

*Of the Division of Compound Algebraic Quantities.*

14. The division of compound algebraic quantities is performed, first, by ranging the several members both of the divisor and dividend according to the dimensions of some letter common to them both, and then proceeding as in common arithmetick.

*N. B.* A quantity is said to be disposed according to the dimensions of any letter in it, when the highest power of that letter is placed first, and the next in order, and so on, as in the following example; where both the divisor and dividend are ranged according to the dimensions of the letter  $x$ .

Of this compound division, take the following example: Let it be required to divide this quantity  $48x^3 - 76ax^2 - 64a^2x + 105a^3$  by this quantity  $2x - 3a$ . Here, as my divisor consists of two members, I take the two first members of the dividend for a first dividual; then I divide the first member of the dividual by the first member of the divisor, to wit,  $48x^3$  by  $2x$ , and the quotient is  $24xx$ , which I put down in the quotient: this done, I multiply the divisor  $2x - 3a$  by the quotient  $24xx$ , and the product is  $48x^3 - 72ax^2$ , which I place under my first dividual  $48x^3 - 76ax^2$ , and then subtracting the former from the latter, I find the remainder to be  $-4ax^2$ ; to this remainder I bring down the next place of my dividend, which is  $-64a^2x$ , and so have a second dividual, to wit,  $-4ax^2 - 64a^2x$ : here again I divide the first member of this dividual, by the first member of the divisor, *viz.*  $-4ax^2$  by  $2x$ , and the quotient is  $-2ax$ , which I put down in the quotient; then multiplying the divisor  $2x - 3a$  by this last quotient  $-2ax$ , the product is  $-4ax^2 + 6a^2x$ , which being subtracted from the second dividual  $-4ax^2 - 64a^2x$  leaves  $-70a^2x$  for a remainder; to this remainder I bring down  $+105a^3$  the last place in the dividend, and so have

have a third dividual  $-70a^2x + 105a^3$ ; the first member whereof, divided by the first member of the divisor, quotes  $-35aa$ , which being put down in the quotient, and the divisor multiplied by it, the product is  $-70a^2x + 105a^3$ , which being subtracted from the last dividual leaves no remainder; so that the whole quotient at last amounts to  $24xx - 2ax - 35aa$ . See the work:

## Example 1st.

$$\begin{array}{r}
 2x-3a \ ) \ 48x^3-76ax^2-64a^2x+105a^3 \ ( \ 24x^2-2ax-35a^2 \\
 \underline{48x^3-72ax^2} \phantom{-64a^2x+105a^3} \\
 * \phantom{2x-3a \ ) \ } -4ax^2-64a^2x \phantom{+105a^3} \\
 \phantom{2x-3a \ ) \ } \underline{-4ax^2+6a^2x} \phantom{+105a^3} \\
 * \phantom{2x-3a \ ) \ } \phantom{-4ax^2} -70a^2x+105a^3 \\
 \phantom{2x-3a \ ) \ } \phantom{-4ax^2} \underline{-70a^2x+105a^3} \\
 \phantom{2x-3a \ ) \ } \phantom{-4ax^2} \phantom{-70a^2x} * \phantom{+105a^3}
 \end{array}$$

## Example 2d.

$$\begin{array}{r}
 4x-5a \ ) \ 48x^3-76ax^2-64a^2x+105a^3 \ ( \ 12x^2-4ax-21aa \\
 \underline{48x^3-60ax^2} \phantom{-64a^2x+105a^3} \\
 * \phantom{4x-5a \ ) \ } -16ax^2-64a^2x \phantom{+105a^3} \\
 \phantom{4x-5a \ ) \ } \underline{-16ax^2+20a^2x} \phantom{+105a^3} \\
 * \phantom{4x-5a \ ) \ } \phantom{-16ax^2} -84a^2x+105a^3 \\
 \phantom{4x-5a \ ) \ } \phantom{-16ax^2} \underline{-84a^2x+105a^3} \\
 \phantom{4x-5a \ ) \ } \phantom{-16ax^2} \phantom{-84a^2x} * \phantom{+105a^3}
 \end{array}$$

This, as I take it, is the most intelligible way of working these divisions, as being the nearest of kin to common division; but it may be somewhat contracted, as will easily be seen by working the foregoing examples over again thus:

## Example 1st.

$$\begin{array}{r}
 2x-3a \ ) \ 48x^3-76ax^2-64a^2x+105a^3 \ ( \ 24x^2-2ax-35a \\
 \underline{48x^3-72ax^2} \\
 * \quad -4ax^2 \\
 \quad \quad -4ax^2+6a^2x \\
 \hline
 \quad \quad \quad -70a^2x \\
 \quad \quad \quad -70a^2x+105a^3 \\
 \hline
 \quad \quad \quad * \quad \quad *
 \end{array}$$

## Example 2d.

$$\begin{array}{r}
 4x-5a \ ) \ 48x^3-76ax^2-64a^2x+105a^3 \ ( \ 12x^2-4ax-21a^2 \\
 \underline{48x^3-60ax^2} \\
 * \quad -16ax^2 \\
 \quad \quad -16ax^2+20a^2x \\
 \hline
 \quad \quad \quad * \quad -84a^2x \\
 \quad \quad \quad \quad -84a^2x+105a^3 \\
 \hline
 \quad \quad \quad * \quad \quad *
 \end{array}$$

This sort of division may be proved the same way as in common division, to wit, by multiplying the divisor into the quotient, or the quotient into the divisor, and adding the remainder, if there be any; for then, if the product, or sum, be equal to the dividend, the division is right, otherwise not: thus if instead of dividing  $48x^3-76ax^2-64a^2x+105a^3$  by  $2x-3a$ , as in the first example, we divide  $48x^3-76ax^2-64a^2x+110a^3$  by  $2x-3a$ , the quotient will be the same as before, to wit,  $24x^2-2ax-35a^2$ , but then there will be a remainder of  $5a^3$ ; therefore, *e converso*, if the quotient  $24x^2-2ax-35a^2$  be multiplied into the divisor  $2x-3a$ , the product, together with the remainder

remainder  $5a^3$ , will make the dividend  $48x^3 - 76ax^2 - 64a^2x + 110a^3$ . See the proof:

$$\begin{array}{r}
 24x^2 - 2ax - 35a^2 \\
 2x - 3a \\
 \hline
 48x^3 - 4ax^2 - 70a^2x + 105a^3 \\
 - 72ax^2 + 6a^2x \\
 \hline
 48x^3 - 76ax^2 - 64a^2x + 105a^3 \\
 + 5a^3 \\
 \hline
 48x^3 - 76ax^2 - 64a^2x + 110a^3.
 \end{array}$$

If, when the divisor and dividend are placed according to the dimensions of some common letter, there be any places wanting, these may be supplied by stars: as if it was required to divide  $16x^4 - 72a^2x^2 + 81a^4$  by  $2x - 3a$ ; there are wanting two places in the dividend, to wit, the places where the third and simple powers of  $x$  are concerned: these therefore being supplied, the dividend will stand thus;  $16x^4 - 72a^2x^2 + 81a^4$ . See the work:

$$\begin{array}{r}
 2x - 3a) 16x^4 - 72a^2x^2 + 81a^4 (8x^3 + 12ax^2 - 18a^2x - 27a^3 \\
 16x^4 - 24ax^3 \\
 \hline
 * + 24ax^3 \\
 + 24ax^3 - 36a^2x^2 \\
 \hline
 * - 36a^2x^2 \\
 - 36a^2x^2 + 54a^3x \\
 \hline
 * - 54a^3x \\
 - 54a^3x + 81a^4 \\
 \hline
 * *
 \end{array}$$

\* \* \*

\* \*

## Of Proportion in Numbers.

15. The rule of proportion in Algebra is so very little different from the rule of proportion in common arithmetick, that one example of it will be sufficient. Let then the following question be put: *If a gives b, what will c give?* Here the second and third terms multiplied together produce  $bc$ ; and the quotient of this, divided by the first term  $a$ , cannot otherwise be expressed than by the fraction  $\frac{bc}{a}$ : this is evident from the notation of fractions explained in the 13th article. But as I have hitherto purposely avoided all consideration of proportion, chusing rather to appeal, upon all occasions, to the common idea every one has or thinks he has of it, than to be more particular, it may not be improper, now we come to reason more closely upon things, to enter more distinctly into the particular nature of proportion, so far at least as it relates to numbers, and shew wherein it consists.

According to *Euclid*, four numbers are said to be proportionable, that is, the first number is said to have the same proportion to the second, that the third hath to the fourth; or the first is said to be to the second, as the third is to the fourth, when the first number is the same multiple, part or parts, of the second, that the third is of the fourth: but it will be asked perhaps, How can we know, what parts, part, or multiple, any one number is of another? To which I answer, by a fraction, whose numerator is the former number, and denominator the latter: thus the fraction  $\frac{2}{3}$  expressly shews, that the numerator 2 is two third parts of the denominator 3; for this is certain, that 1 is  $\frac{1}{3}$  part of 3, and therefore 2 must be  $\frac{2}{3}$  of it: for the same reason the fraction  $\frac{12}{8}$  shews that the number 12 is  $\frac{12}{8}$  or  $\frac{3}{2}$  of the number 8; and lastly, the fraction  $\frac{12}{8}$  shews that the number

ber 12 is  $\frac{1}{4}$  of, or 3 times the number 4; and consequently, that 12 is a multiple of 4, as containing it just 3 times without any remainder: therefore, to any one who understands fractions, *Euclid's* definition of proportion may be more distinctly expressed thus: *Four numbers are said to be proportionable, when a fraction whose numerator is the first number, and denominator the second, is equal to a fraction whose numerator is the third number, and denominator the fourth.* Thus 2 is to 3 as 4 is to 6, because  $\frac{2}{3}$  is equal to  $\frac{4}{6}$ ; thus 12 is to 8 as 15 is to 10, because  $\frac{12}{8}$  equals  $\frac{15}{10}$ , both being reducible to  $\frac{3}{2}$ ; thus 2 is to 6 as 4 is to 12, because  $\frac{2}{6}$  equals  $\frac{4}{12}$ , for each is equal to  $\frac{1}{3}$ ; lastly, 6 is to 2 as 12 is to 4, because  $\frac{6}{2} = \frac{12}{4} = 3$ .

From this idea of proportionality may be demonstrated a very useful theorem in Algebra; which is, that *Whenever four numbers are proportionable, the product of the extreme terms multiplied together will be equal to the product of the two middle terms so multiplied*: for let  $a, b, c$ , and  $d$ , be four proportionable numbers in their order; that is, let  $a$  be to  $b$  as  $c$  is to  $d$ ; I say then that  $ad$  the product of the extremes will be equal to  $bc$  the product of the two middle terms: for since  $a$  is to  $b$  as  $c$  is to  $d$ , it follows, from what has already been laid down, that the fraction  $\frac{a}{b}$  is equal

to the fraction  $\frac{c}{d}$ ; multiply both of the terms of the

fraction  $\frac{a}{b}$  into  $d$ , and both those of the fraction

$\frac{c}{d}$  into  $b$  (which multiplications may be made without altering the values of the fractions), and then you

will have  $\frac{ad}{bd} = \frac{bc}{bd}$ ; that is, the quotient of  $ad$  divided by  $bd$ , is equal to the quotient of  $bc$  divided by  $bd$ ; therefore  $ad$  must be equal to  $bc$ , that is, the

the product of the extremes must be equal to the product of the middle terms. *Q. E. D.*

The converse of this proposition is also true, to wit, that *Whenever we have an equation in numbers, wherein the product of two numbers on one side is found equal to the product of two numbers on the other, such an equation may be resolved into four proportionals, by making the two numbers on either side, the extremes; and those on the other side, the middle terms: thus if  $ad=bc$ ; by making  $a$  and  $d$  the extremes, and  $b$  and  $c$  the middle terms, we shall have  $a$  to  $b$  as  $c$  to  $d$ : if this be denied, let  $a$  be to  $b$  as  $c$  to  $e$ ; then we shall have  $ae=bc$  by the last; but  $ad=bc$  by the supposition; therefore  $ae=ad$ ; therefore  $e$  equals  $d$ , and  $a$  is to  $b$  as  $c$  is to  $d$ . *Q. E. D.**

#### C O R O L L A R Y.

Whence if  $a$ ,  $b$ , and  $c$ , be continual proportionals, that is, if  $a$  is to  $b$  as  $b$  is to  $c$ , we shall have  $b^2=ac$ : and, *è converso*, if  $b^2=ac$ , then  $a$ ,  $b$ , and  $c$ , will be continual proportionals.

*The common Properties of Proportionality in Numbers demonstrated.*

16. From what has been delivered in the last article may be demonstrated all, or most of the common properties of proportionable numbers with a great deal of ease; some of the most useful whereof I shall here throw together into one single article, for the reader to peruse, either at present, or hereafter, as he shall see occasion.

First then, from what has been said, may the rule of three, which consists in finding a fourth proportional, be most distinctly demonstrated: for let  $a$ ,  $b$ , and  $c$ , be three numbers given, in order to find  $d$ , a fourth proportional; then since  $a$  is to  $b$  as  $c$  is to  $d$ , you will have  $ad$ , the product of the extremes, equal  
to

to  $bc$ , the product of the middle terms; divide both sides of the equation by  $a$ , and you will have  $d = \frac{bc}{a}$ : which is as much as to say, that if three numbers be given, a fourth proportional may be obtained by multiplying the second and third numbers together, and dividing the product by the first.

In the rule of three inverse, let the numbers, when disposed according to form, be  $a$ ,  $b$ , and  $c$ ; then whoever attentively considers the nature of that rule, will easily see, that the fourth number there sought for is not to be a fourth proportional to the three numbers given as they are disposed in the order  $a$ ,  $b$ ,  $c$ , but as they stand in the order  $c$ ,  $b$ ,  $a$ , or  $c$ ,  $a$ ,  $b$ , and therefore, in this case, the fourth number will be  $\frac{ab}{c}$ .

Secondly, if two proportions be equal to a third, they must be equal to one another, because if two fractions be equal to a third, they must be equal to one another: thus if  $a$  is to  $b$  as  $c$  is to  $d$ , and  $c$  is to  $d$  as  $e$  is to  $f$ , we shall have  $a$  to  $b$  as  $e$  to  $f$ .

Thirdly, if  $a$  is to  $b$  as  $c$  is to  $d$ ; then  $b$  will be to  $a$  as  $d$  to  $c$ , which is called inverse proportion: for if  $a$  is to  $b$  as  $c$  is to  $d$ , we shall have  $ad = bc$ ; make  $b$  and  $c$  the extremes, and you will have  $b$  to  $a$  as  $d$  to  $c$ .

Fourthly, if  $a$  is to  $b$  as  $c$  is to  $d$ ; we shall have, by permutation,  $a$  to  $c$  as  $b$  to  $d$ : for since  $a$  is to  $b$  as  $c$  is to  $d$ , and consequently  $ad = bc$ , make  $a$  and  $d$  the extremes, and  $c$  and  $b$  the middle terms, and you will have  $a$  to  $c$  as  $b$  to  $d$ .

Fifthly, if  $a$  is to  $b$  as  $c$  is to  $d$ , and any two multipliers whatever be assumed, as  $e$  and  $f$ ; I say then, that  $ea$  is to  $fb$  as  $ec$  is to  $fd$ : for since  $a$  is to  $b$  as  $c$  is to  $d$ , and so  $ad = bc$ ; multiply both sides of the equation by the product  $ef$ , and you will have

have  $ad \times ef = bc \times ef$ ; but  $ad \times ef = ea \times fd$ , and  $bc \times ef = fb \times ec$ ; therefore  $ea \times fd = fb \times ec$ ; make  $ea$  and  $fd$  extremes, and the proportion will stand thus;  $ea$  is to  $fb$  as  $ec$  to  $fd$ . In like manner, *mutatis mutandis*, it may be demonstrated, that if  $a$  is to  $b$  as  $c$  is to  $d$ ; then  $\frac{a}{e}$  will be to  $\frac{b}{f}$  as  $\frac{c}{e}$  is to  $\frac{d}{f}$ .

Sixthly, if  $a$  is to  $b$  as  $c$  is to  $d$ ; then  $a^2$  is to  $b^2$  as  $c^2$  is to  $d^2$ : for since  $a$  is to  $b$  as  $c$  is to  $d$ , and so  $ad = bc$ ; square both sides of the equation, and you will have  $a^2 d^2 = b^2 c^2$ ; make  $a^2$  and  $d^2$  extremes, and you will have  $a^2$  to  $b^2$  as  $c^2$  to  $d^2$ . And by taking these steps backwards, it will also appear, that if  $a^2$  is to  $b^2$  as  $c^2$  is to  $d^2$ ;  $a$  is to  $b$  as  $c$  is to  $d$ , and  $\sqrt{a}$  is to  $\sqrt{b}$  as  $\sqrt{c}$  is to  $\sqrt{d}$ .

Seventhly, if  $a$  is to  $b$  as  $c$  is to  $d$ ; then by composition (as it is called)  $\overline{a+b}$  is to  $b$  as  $\overline{c+d}$  is to  $d$ ; or  $\overline{a+b}$  is to  $a$  as  $\overline{c+d}$  is to  $c$ : for since  $a$  is to  $b$  as  $c$  is to  $d$ , and consequently  $ad = bc$ ; add  $bd$  to both sides of the equation, and you will have  $\overline{ad+bd} = \overline{bc+bd}$ ; but  $\overline{ad+bd}$  is the product of  $\overline{a+b}$  multiplied into  $d$ , as is easily seen; and  $\overline{bc+bd}$  is the product of  $b$  multiplied into  $\overline{c+d}$ ; therefore  $\overline{a+b} \times d = b \times \overline{c+d}$ ; make  $\overline{a+b}$  and  $d$  extremes, and you will have  $\overline{a+b}$  to  $b$  as  $\overline{c+d}$  to  $d$ . Again, since  $bc = ad$ , add  $ac$  to both sides, and you will have  $\overline{ac+bc} = \overline{ac+ad}$ , that is,  $\overline{a+b} \times c = a \times \overline{c+d}$ ; make  $\overline{a+b}$  and  $c$  extremes, and you will have  $\overline{a+b}$  to  $a$  as  $\overline{c+d}$  to  $c$ .

Eighthly, if  $a$  is to  $b$  as  $c$  is to  $d$ ; then by division  $\overline{a-b}$  is to  $b$  as  $\overline{c-d}$  is to  $d$ ; or  $\overline{a-b}$  is to  $a$  as  $\overline{c-d}$  to  $c$ . This proposition is demonstrated by subtraction, just in the same manner as the last was by addition.

Ninthly, if to or from two numbers in any given proportion, be added or subtracted other two numbers

in the same proportion, the sums or remainders will still be in the same proportion with the numbers first proposed: thus if the numbers  $c$  and  $d$  be in the same proportion with the numbers  $a$  and  $b$ , that is, if as  $a$  is to  $b$  so is  $c$  to  $d$ , and if to or from the former two numbers be added or subtracted the latter, we shall have not only  $\overline{a+c}$  to  $\overline{b+d}$  as  $a$  to  $b$ , but also  $\overline{a-c}$  to  $\overline{b-d}$  as  $a$  to  $b$ : for since, by the supposition,  $a$  is to  $b$  as  $c$  is to  $d$ ; it follows by permutation, that  $a$  is to  $c$  as  $b$  is to  $d$ ; and by composition that  $\overline{a+c}$  is to  $a$  as  $\overline{b+d}$  to  $b$ ; and again by permutation, that  $\overline{a+c}$  is to  $\overline{b+d}$  as  $a$  is to  $b$ : in like manner by permutation and division we shall have  $\overline{a-c}$  to  $\overline{b-d}$  as  $a$  to  $b$ .

Tenthly, if there be three numbers  $a$ ,  $b$ , and  $c$ , and other three numbers  $d$ ,  $e$ ,  $f$ , proportionable to them, and in the same order, that is, if as  $a$  is to  $b$  so  $d$  is to  $e$ , and as  $b$  is to  $c$  so is  $e$  to  $f$ ; I say then, that, *ex æquo*, the extremes will be in the same proportion, (*viz.*) that  $a$  will be to  $c$  as  $d$  is to  $f$ : for since by the supposition,  $a$  is to  $b$  as  $d$  is to  $e$ ; by permutation we shall have  $a$  to  $d$  as  $b$  to  $e$ ; and for the same reason, since  $b$  is to  $c$  as  $e$  is to  $f$ ; we shall have  $b$  to  $e$  as  $c$  to  $f$ : since then  $a$  is to  $d$  as  $b$  to  $e$ , and  $b$  to  $e$  as  $c$  to  $f$ ; it follows from the second proposition, that  $a$  is to  $d$  as  $c$  to  $f$ ; and by permutation, that  $a$  is to  $c$  as  $d$  to  $f$ .

Eleventhly, if there be three numbers,  $a$ ,  $b$ , and  $c$ , and three other numbers  $d$ ,  $e$ , and  $f$ , proportionable to them, but in a contrary order, so that  $a$  is to  $b$  as  $e$  to  $f$ , and  $b$  to  $c$  as  $d$  to  $e$ ; I say, that the extremes will still be proportionable, to wit, that  $a$  will be to  $c$  as  $d$  to  $f$ : for since  $a$  is to  $b$  as  $e$  to  $f$ , we have  $af=be$ ; moreover, since  $b$  is to  $c$  as  $d$  to  $e$ , we have  $cd=be$ ; therefore  $af=cd$ ; make  $a$  and  $f$  extremes, and you will have  $a$  to  $c$  as  $d$  to  $f$ .

N. B. If there be two serieses of numbers, as  $a$ ,  $b$ ,  $c$ , &c.;  $d$ ,  $e$ ,  $f$ , &c.; each series consisting of the same number

number of terms; and if all the proportions between contiguous terms in one series be respectively equal to all those in the other, that is, each to each, as they stand in order; as if  $a$  be to  $b$  as  $d$  to  $e$ , and  $b$  to  $c$  as  $e$  to  $f$ , &c.; then the extreme terms of one series will be proportionable to the extreme terms of the other: for the demonstration of the tenth proposition may be extended to as many terms as we please; and this proportionality of the extremes is said to follow *ex æquo ordinate*, or barely *ex æquo*, that is, from a respective equality of all the proportions in one series to their correspondents in the other, in an orderly manner. But if every proportion in one series has an equal proportion to answer it in the other, but not in a correspondent part of the series; as if  $a$  be to  $b$  as  $e$  to  $f$ , and  $b$  to  $c$  as  $d$  to  $e$ , &c.; then though the extremes will still be proportionable, as will be evident by continuing the demonstration of this eleventh proposition; yet now the proportionality of the extremes is said to follow *ex æquo perturbate*, that is, from an equality of all the proportions in one series to all those in the other, but in a disorderly manner.

Twelfthly, if  $a$  is to  $b$  as  $c$  is to  $d$ ; we shall have  $\overline{a+b}$  to  $\overline{a-b}$  as  $\overline{c+d}$  is to  $\overline{c-d}$ : for since  $a$  is to  $b$  as  $c$  is to  $d$ , we shall have by composition  $\overline{a+b}$  to  $\overline{a}$  as  $\overline{c+d}$  is to  $\overline{c}$ ; we shall have also by division,  $\overline{a-b}$  to  $\overline{a}$  as  $\overline{c-d}$  to  $\overline{c}$ ; and by inversion,  $a$  to  $\overline{a-b}$  as  $c$  to  $\overline{c-d}$ : since then we have  $\overline{a+b}$  to  $\overline{a}$  as  $\overline{c+d}$  to  $\overline{c}$ ; and  $a$  to  $\overline{a-b}$  as  $c$  to  $\overline{c-d}$ , that is, since we have three numbers,  $\overline{a+b}$ ,  $a$ , and  $\overline{a-b}$ , and other three numbers proportionable to them in the same order, to wit,  $\overline{c+d}$ ,  $c$ , and  $\overline{c-d}$ ; it follows *ex æquo* that the extremes will be proportionable, that is, that  $\overline{a+b}$  will be to  $\overline{a-b}$  as  $\overline{c+d}$  is to  $\overline{c-d}$ .

Thirteenthly, if there be a series of numbers,  $k$ ,  $l$ ,  $m$ ,  $n$ , whereof  $k$  is to  $l$  as  $a$  to  $b$ , and  $l$  to  $m$  as  $c$

to  $d$ , and  $m$  to  $n$  as  $e$  to  $f$ ; I say then that  $k$  the first term will be to  $n$  the last, as  $ace$  the product of all the other antecedents to  $bdf$  the product of all the other consequents; for  $k$  is to  $l$  as  $l$  to  $b$ , by the supposition; and we shall find that  $a$  is to  $b$  as  $ace$  to  $bce$  by multiplying extremes and means; therefore  $k$  is to  $l$  as  $ace$  to  $bce$ ; and for a like reason  $l$  is to  $m$  as  $bce$  to  $bde$ , and  $m$  is to  $n$  as  $bde$  to  $bdf$ ; therefore, *ex aequo*,  $k$  is to  $n$  as  $ace$  to  $bdf$ .

*Of the Extraction of the Square Roots of simple Algebraic Quantities.*

17. The extraction of the square root of simple algebraic quantities is so very easy that it needs not to be insisted on. Thus the square root of  $aa$  is  $+$  or  $-a$ , the square root of  $9aa$  is  $+$  or  $-3a$ , and that of  $4aabb$  is  $+$  or  $-2ab$ : this is plain from the definition of the square root; for the square root of any quantity, suppose of  $4aabb$ , is that which, being multiplied into itself, will produce  $4aabb$ : now  $-2ab$  multiplied into itself will produce  $4aabb$ , as well as  $+2ab$ , and therefore one quantity is as much its square root as the other.

When the square root of a quantity cannot be extracted, it is usual to signify it by this mark  $\sqrt{\phantom{x}}$ : thus  $\sqrt{2aa}$  signifies the square root of  $2aa$ ; thus  $\sqrt{aa-4b}$  signifies the square root of the whole quantity  $aa-4b$ ; thus  $\frac{\sqrt{aa-4b}}{2a}$  signifies a fraction whose numerator is

the square root of the whole quantity  $aa-4b$ , and whose denominator is  $2a$ ; thus  $\sqrt{\frac{4ab-a^3}{12a}}$  signifies

the square root of the whole fraction,  $\frac{4ab - a^3}{12a}$ , that is, the square root of both the numerator and denominator.

*When the square root of a quantity cannot be extracted, the quantity may sometimes however be resolved into two factors, whereof the one is a square, and the other is not; and whenever this is possible, the root of the square may be extracted, and the radical sign may be prefixed to the other factor; thus  $12a = 4aa \times 3$ ; therefore  $\sqrt{12aa} = 2a \times \sqrt{3}$ .*

*The several Rules of Fractions exemplified in Algebraic Quantities.*

22. Fractions in Algebra are treated just in the same manner as in common arithmetic, only using algebraical instead of numeral operations; as will plainly appear from the following examples.

*Examples of the Reduction of Fractions from higher to lower Terms, according to Introduction, art. 7th.*

The fraction  $\frac{4ab}{6bc}$ , dividing both the numerator and denominator by the same quantity  $2b$ , will be reduced to the fraction  $\frac{2a}{3c}$ , a fraction of the same value with the former, but expressed in more simple terms: whence we may infer, that *whenever a common letter or factor is to be found in every member both of the numerator and denominator, it may be cancelled everywhere, without affecting the value of the fraction*; thus the fraction  $\frac{ac+bc}{cd+ce}$ , expunging  $c$ , becomes  $\frac{a+b}{d+e}$ , a fraction of the same value. But if there be any one member wherein the factor is not concerned, it must

not be expunged at all : thus the fraction  $\frac{ac+bc}{cd+e}$  cannot be reduced, because the factor  $c$  is not to be found in  $e$ .

*Note, That cancelling here, is not subtracting, but dividing :* thus to cancel the letter  $b$  in the quantity  $ab$ , so as to reduce it to  $a$ , is not to subtract  $b$  from  $ab$ , but to divide  $ab$  by  $b$ , in which case the quotient will be  $a$ .

*Examples of Fractions reduced to the same Denomination, according to Introduction, art. 8th.*

1st. The fractions  $\frac{a}{2}$ ,  $\frac{b}{3}$ , and  $\frac{c}{4}$ , when reduced to the same denomination, will stand thus ;  $\frac{12a}{24}$ ,  $\frac{8b}{24}$ , and  $\frac{6c}{24}$

2d. The fractions  $\frac{a}{b}$  and  $\frac{c}{d}$ , so reduced, will stand

thus ;  $\frac{ad}{bd}$  and  $\frac{bc}{bd}$ . 3d. The fractions  $\frac{p}{q}$ ,  $\frac{r}{s}$ ,  $\frac{t}{u}$ , and  $\frac{x}{y}$ ,

after reduction, will stand thus ;  $\frac{psuy}{qsuy}$ ,  $\frac{gruy}{qsuy}$ ,  $\frac{qsty}{qsuy}$ ,

and  $\frac{qsux}{qsuy}$ . And here I cannot but observe, that now

the rule for this reduction demonstrates itself : for in this example it is impossible not to see, that all these fractions, notwithstanding this reduction, still retain

their former values : thus the first fraction  $\frac{psuy}{qsuy}$ , by

cancelling common factors, is reduced to  $\frac{p}{q}$ , its former value : and the same may be observed of all the

rest : and this example amounts to a demonstration, because it is comprehended in general terms. But to go

on ;

on: 4th. The fractions  $\frac{1}{a}$ ,  $\frac{1}{b}$ , and  $\frac{1}{c}$ , being reduced

to the same denomination, become  $\frac{bc}{abc}$ ,  $\frac{ac}{abc}$ , and  $\frac{ab}{abc}$ .

5th. And lastly,  $\frac{1}{a+b}$  and  $\frac{1}{a-b}$  when thus reduced,

become  $\frac{a-b}{aa-bb}$  and  $\frac{a+b}{aa-bb}$ : for 1 the numerator of the first fraction multiplied into  $a-b$ , the denominator of the second, makes  $a-b$ ; and 1 the numerator of the second fraction multiplied into  $a+b$ , the denominator of the first, makes  $a+b$ ; and the product of the two denominators  $a+b$  and  $a-b$  multiplied together is  $aa-bb$ , as in the 4th example of the 9th article.

*Examples of Addition in Fractions, according to Introduction, art. 9th.*

1st. These fractions  $\frac{a}{2}$ ,  $\frac{b}{2}$ , and  $\frac{-c}{2}$ , when added together, make  $\frac{a+b-c}{2}$ .

2d. The fraction  $\frac{a+b}{2}$  added to the fraction  $\frac{a-b}{2}$  makes  $\frac{2a}{2}$  or  $a$ .

3d. The fractions  $\frac{a}{2}$ ,  $\frac{-b}{3}$  and  $\frac{+c}{4}$ , when added together, make  $\frac{12a-8b+6c}{24}$ .

4th. The fraction  $\frac{a}{b}$  added to the fraction  $\frac{c}{d}$  makes  $\frac{ad+bc}{bd}$ .

5th.  $a$  added to  $\frac{b}{c}$ , that is,  $\frac{a}{1}$  added to  $\frac{b}{c}$ , makes  $\frac{ac+b}{c}$ .

6th.  $\frac{1}{a}$  added to  $-\frac{1}{b}$  makes  $\frac{b-a}{ab}$ .

7th. The fractions  $\frac{p}{q}$ ,  $\frac{r}{s}$ ,  $\frac{t}{u}$ , and  $\frac{x}{y}$ , when added together, make  $\frac{psuy+qrux+qsty+qsux}{qsuy}$ .

8th.  $\frac{a}{b}$  added to  $\frac{1}{c}$  gives  $\frac{ac+1}{bc}$ .

9th.  $\frac{1}{a+b}$  added to  $\frac{1}{a-b}$  gives  $\frac{2a}{aa-bb}$ . See the 5th example of fractions reduced to the same denomination.

*Examples of Subtraction in Fractions, according to Introduction, art. 10th.*

*Note first,* If the signs of both the numerator and denominator of any fraction be changed, which is no more than multiplying both terms into  $-1$ , the value of the fraction will still remain.

*Secondly,* The denominator of a fraction is always supposed to be affirmative; and therefore if at any time it happens to be otherwise, it must be made affirmative by changing the signs of both terms.

*Thirdly,*  $+\frac{a}{b}$  and  $-\frac{a}{b}$  are the same in effect as  $\frac{+a}{b}$  and  $\frac{-a}{b}$ , as is evident from the nature of division: and sometimes this latter way of notation is more convenient than the former.

*Fourthly,*

*Fourthly*, Therefore the sign of the numerator is the sign of the whole fraction; and to change the sign of the former, is the same in effect as to change the sign of the latter.

*Fifthly*, Whenever one algebraic fraction is to be subtracted from another, the safest way will be to change the sign of the numerator of the fraction to be subtracted, and to place it after the other, and then to reduce them at last into one fraction: for if the subtraction be deferred till after the reduction is over, one may make a mistake, and subtract the wrong quantity. Thus, 1st,  $\frac{4b}{5}$  subtracted from  $\frac{2a}{3}$  gives

$$\frac{2a}{3} - \frac{4b}{5} = \frac{10a - 12b}{15}.$$

$$2d. \frac{r}{s} \text{ subtracted from } \frac{p}{q} \text{ gives } \frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs}.$$

$$3d. \frac{b}{c} \text{ subtracted from } a \text{ gives } \frac{a}{1} - \frac{b}{c} = \frac{ac - b}{c}.$$

$$4th. \frac{1}{a+b} \text{ subtracted from } \frac{1}{a-b} \text{ gives } \frac{1}{a-b} - \frac{1}{a+b} = \frac{2b}{aa - bb}.$$

### *Examples of Multiplication in Fractions.*

The multiplication of fractions is performed by multiplying the numerator and denominator of the multiplicand into the numerator and denominator of the multiplier respectively.

$$\text{Thus 1st. } \frac{p}{q} \times \frac{r}{s} = \frac{pr}{qs}.$$

$$2d. \frac{3p}{4q} \times \frac{5q}{6r} = \frac{15pq}{24qr} = \frac{5p}{8r}.$$

$$3d. \frac{a}{b} \times c \text{ or } \frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}.$$

4th.

$$4\text{th. } \frac{a}{b} \times b = \frac{ab}{b} = a.$$

$$5\text{th. } \frac{3a}{4b} \times 20b = \frac{60ab}{4b} = 15a.$$

$$6\text{th. } \frac{4a}{5} \times \frac{7}{8a} = \frac{28a}{40a} = \frac{7}{10}.$$

$$7\text{th. } \frac{3a}{4b} \times \frac{3a}{4b} = \frac{9aa}{16bb}.$$

$$8\text{th. } \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}.$$

$$9\text{th. } a + \frac{b}{c} \times d, \text{ or } \frac{ac+b}{c} \times \frac{d}{1} = \frac{acd+bd}{c}.$$

$$10\text{th. } d + \frac{e}{f} \times \frac{g}{b}, \text{ or } \frac{df+e}{f} \times \frac{g}{b} = \frac{dfg+eg}{fb}.$$

$$11\text{th. } a + \frac{b}{c} \times d + \frac{e}{f}, \text{ or } \frac{ac+b}{c} \times \frac{df+e}{f} = \frac{acdf+ace+ bdf+be}{cf}.$$

This multiplication might also be performed thus :

$$d + \frac{e}{f}$$

$$a + \frac{b}{c}$$

$$ad + \frac{ae}{f} + \frac{bd}{c} + \frac{be}{cf}.$$

$$12\text{th. } a + \frac{b}{c} \times a + \frac{b}{c} = \frac{aacc+2abc+bb}{cc} :$$

$$\text{Or, } aa + \frac{2ab}{c} + \frac{bb}{cc}. \text{ See the work :}$$

$$a +$$

$$\begin{array}{r}
 a + \frac{b}{c} \\
 a + \frac{b}{c} \\
 \hline
 aa + \frac{ab}{c} + \frac{bb}{cc} \\
 + \frac{ab}{c} \\
 \hline
 aa + \frac{2ab}{c} + \frac{bb}{cc}
 \end{array}$$

*Examples of Division in Fractions.*

Division in fractions is performed by multiplying the direct terms of the dividend into the inverted terms of the divisor ; thus,

$$1^{\text{st}}. \frac{r}{s} \Big) \frac{p}{q} \left( \frac{ps}{qr}. \qquad 2^{\text{d}}. \frac{b}{c} \Big) \frac{1}{a} \left( \frac{c}{ab}.$$

$$3^{\text{d}}. \frac{1}{c} \Big) \frac{a}{b} \left( \frac{ac}{b}. \qquad 4^{\text{th}}. c \Big) \frac{a}{b} \left( \frac{a}{bc}.$$

$$5^{\text{th}}. \frac{1}{c} \Big) \frac{ab}{c} \left( \frac{abc}{c}, \text{ or } ab.$$

$$6^{\text{th}}. d + \frac{e}{f} \Big) a + \frac{b}{c} \left( \frac{acf + bf}{cdf + ce}.$$

$$7^{\text{th}}. \sqrt{b} \Big) \sqrt{a} \left( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} : \text{ for if we make } x =$$

$$\frac{\sqrt{a}}{\sqrt{b}}, \text{ we shall have } xx = \frac{a}{b}, \text{ and } x = \sqrt{\frac{a}{b}}.$$

I shall

I shall only give one example more, and that shall be of the rule of proportion, as follows: If  $\frac{a}{b}$  gives  $\frac{c}{d}$ , what will  $\frac{e}{f}$  give? Answer,  $\frac{bce}{adf}$ : for  $\frac{c}{d}$  the second number, multiplied into  $\frac{e}{f}$  the third, produces  $\frac{ce}{df}$ ; and this divided by the first  $\frac{a}{b}$  quotes  $\frac{bce}{adf}$ .

*Of Equations in Algebra; and particularly of simple Equations, together with the Manner of resolving them.*

23. *An equation in Algebra is a proposition wherein one quantity is declared equal to another, or where one expression of any quantity is declared equal to another expression of the same quantity: as when we say  $\frac{2}{4} = \frac{3}{6}$ ; where  $\frac{2}{4}$  is said to possess one side of the equation, and  $\frac{3}{6}$  the other.*

An affected quadratic equation is an equation consisting of three different sorts of quantities; one wherein the square of the unknown quantity is concerned, another wherein the unknown quantity is simply concerned, and a third wherein it is not concerned at all: as if  $xx - 2x = 3$ ; supposing  $x$  to be an unknown quantity.

If either the term wherein the simple power of  $x$  is concerned, as  $-2x$ , or that which is called the absolute term, to wit, 3, be wanting, the equation is still a quadratic equation, though incomplete. Some indeed there are, who rank this latter sort of equations under the denomination of simple equations; and so shall we, upon account of their easy resolution; though, properly speaking, a simple equation is that wherein some simple power of the  
unknown

unknown quantity is concerned, all others being excluded: as if  $3x=6$ ;  $2x+3=4x-5$ , &c.

*The use of these equations is for representing more conveniently and more distinctly the conditions of problems, when translated out of common language into that of Algebra.* As for example; let it be proposed to find a number with the following property, to wit, that  $\frac{2}{3}$  of it with 4 over may amount to the same as  $\frac{7}{12}$  of it with 9 over: here, putting  $x$  for the unknown quantity, the condition of this problem, when translated out of common language into that of Algebra, will be represented by the following equation, to wit,  $\frac{2x}{3}+4=\frac{7x}{12}+9$ : for  $\frac{2}{3}$  of  $x$ , that

is,  $\frac{2}{3}$  of  $\frac{x}{1}$  is  $\frac{2x}{3}$ ; therefore  $\frac{2x}{3}+4$  signifies  $\frac{2}{3}$  of  $x$  with 4 over: and since this expression, according to the problem, amounts to the same with the other, to wit,  $\frac{7x}{12}+9$ ; hence it is that we pronounce them equal to one another.

Now since, *in the foregoing equation*, as well as in almost all others arising immediately from the conditions of problems themselves, *the unknown quantity is embarrassed and entangled with such as are known, the way to disengage it from such known quantities, so that itself alone possessing one side of the equation may be found equal to such as are entirely known on the other*, that is, in the present case, to determine the value of the unknown quantity  $x$ , is what is commonly called *the resolution of an equation*: for the effecting of which, several axioms and processes are required; some whereof, namely, such as most frequently occur, I shall here put down; the rest I shall take notice of occasionally, as they offer themselves.

*Of the Resolution of simple Equations.*

## A X I O M 1.

Whenever a fraction is to be multiplied by a whole number, it will be sufficient to multiply only the numerator by that number, retaining the denominator the same as before. Thus  $\frac{4}{5}$  multiplied into 2, gives  $\frac{8}{5}$ , for the same reason that 4 shillings multiplied into 2 gives 8 shillings: thus in the first example following,  $\frac{7x}{12}$  multiplied into 3, gives  $\frac{21x}{12}$ .

## A X I O M 2.

But if the whole number, into which the fraction is to be multiplied, be equal to the denominator of the fraction, then throw away the denominator, and the numerator alone will be the product. Thus the fraction  $\frac{a}{b}$  multiplied into  $b$ , gives  $\frac{ab}{b}$  or  $a$ : thus in the first example,  $\frac{2x}{3}$  multiplied into 3, gives  $2x$ ; and  $\frac{21x}{12}$  multiplied into 12, gives  $21x$ .

## A X I O M 3.

If the two sides of an equation be multiplied or divided by the same number, the two products, or quotients, will still be equal to each other. Thus in the first example, where  $\frac{2x}{3} + 4 = \frac{7x}{12} + 9$ ; if both sides of the equation be multiplied into 3, we shall have  $2x + 12 = \frac{21x}{12} + 27$ ; and if again this last equation be multiplied into 12, we shall have  $24x + 144 = 21x + 324$ .

AXIOM

## Axiom 4.

*If a quantity be taken from either side of an equation, and placed on the other with a contrary sign, which is commonly called transposition, the two sides will be equal to each other.* Thus if  $7 + 3 = 10$ , transpose  $+3$ , and you will have  $7 = 10 - 3$ ; thus if  $7 - 3 = 4$ , transpose  $-3$ , and you will have  $7 = 4 + 3$ : thus if (as in the first example)  $24x + 144 = 21x + 324$ , transpose  $21x$ , and you will have  $24x - 21x + 144 = 324$ , that is,  $3x + 144 = 324$ ; and if again in this last equation you transpose  $144$ , you will have  $3x = 324 - 144 = 180$ .

Transposition, therefore, as it is here delivered, is nothing but a general name for adding or subtracting equal quantities from the two sides of an equation; in which case it is no wonder if the sums or differences still continue equal to each other. As for instance, in this equation  $a - b = c$ , transposing  $-b$  we have  $a = c + b$ : and what is this, after all, but adding  $b$  to both sides of the equation? for if  $b$  be added to  $a - b$ , the sum will be  $a$ ; and if  $b$  be added to  $c$ , the sum will be  $c + b$ ; therefore  $a = c + b$ : again in the equation  $a + b = c$ , transposing  $+b$ , we have  $a = c - b$ , which is nothing else but subtracting  $b$  from both sides of the equation.

## The 1st Process.

*If, when an equation is to be resolved, fractions be found on one or both sides, it must be freed from them by multiplying the whole equation into the denominators of those fractions successively.*

## The 2d Process.

*After the equation is thus reduced to integral terms, if the unknown quantity be found on both sides the equation, let it be brought by transposition to one and the same side, viz. to that side which after reduction will exhibit it affirmative.*

The

## The 3d Process.

*After this, if any loose known quantities be found on the same side with the unknown, let them also be brought by transposition to the other side of the equation.*

## The 4th Process.

*If now the unknown quantity has any coefficient before it, divide all by that coefficient, and the equation will be resolved.*

## The 5th Process.

*If the whole equation can be divided by the unknown quantity, let such a division be made, and the equation will be reduced to a more simple one. Thus in the 16th example you have  $615x - 7xxx = 48x$ ; divide the whole equation by  $x$ , and you will have  $615 - 7xx = 48$ .*

*In the 13th example you have  $\frac{42x}{x-2} = \frac{35x}{x-3}$ ; divide the whole by  $x$ , which is done by dividing only the numerators of the two fractions, and you will have*

$$\frac{42}{x-2} = \frac{35}{x-5}.$$

## The 6th Process.

*If at last the square of the unknown quantity, and not the unknown quantity itself, appears to be equal to some known quantity on the other side of the equation, then the unknown quantity must be made equal to the square root of that which is known. Thus in the 14th example we have  $xx = 36$ ; therefore  $x = 6$ , and not 18: in the 15th, we have  $xx = 64$ ; therefore  $x = 8$ , the square root of 64, and not 32, its half.*

*Examples of the Resolution of simple Equations.*

24. This preparation being made, I shall now give some examples of the resolution of simple equations; and

and my first example shall be the equation given in the last article, in order to trace out the number there described.

Example 1.

$$\frac{2x}{3} + 4 = \frac{7x}{12} + 9.$$

In this equation it is plain, that there are two fractions,  $\frac{2x}{3}$ , and  $\frac{7x}{12}$ , which must be taken off at two several operations, thus : as 3 is the denominator of the first fraction, multiply the whole equation by 3, and you will have  $2x + 12 = \frac{21x}{12} + 27$  : again, as the denominator of the remaining fraction is 12, multiply all by 12, and you will have  $24x + 144 = 21x + 324$ ; which is an equation free from fractions.

2dly, It must in the next place be considered, that in this last equation  $24x + 144 = 21x + 324$ , the unknown quantity is concerned on both sides, to wit,  $24x$  on one side, and  $21x$  on the other; transpose therefore  $21x$ , and you will have  $24x - 21x + 144 = 324$ , that is,  $3x + 144 = 324$ . If it be asked why I chose to transpose  $21x$  rather than  $24x$ ; my answer is, that had  $24x$  been transposed, the unknown quantity, or its coefficient at least, after reduction, would have been negative, contrary to the rule in the second process; for, resuming the equation  $24x + 144 = 21x + 324$ , if  $24x$  be transposed, we shall have  $144 = 21x - 24x + 324$ ; that is,  $144 = -3x + 324$ : but even in this case, another transposition will set all right; for if  $-3x$  be transposed in this last equation, we shall then have  $3x + 144 = 324$  as before: all that can be said then against this last way is, that it creates unnecessary transpositions, which an artist would always endeavour to avoid.

3dly, Having now reduced the equation to a much greater degree of simplicity than before, to wit,  $3x + 144 = 324$ ; because the unknown quantity  $3x$  has

still a loose quantity, *viz.* 144 joined with it, transpose that quantity 144 to the other side of the equation, and you will have  $3x = 324 - 144$ , that is,  $3x = 180$ .

*N. B.* By a loose quantity I mean such a one as is joined with the unknown by the sign  $+$  or  $-$ , and not by way of multiplication, as is the coefficient 3 in the last equation.

4<sup>thly</sup>, By this time the quantity  $x$  is very near being discovered: for if  $3x = 180$ , it is but dividing all by 3, and we shall have  $x = 60$ : 60 therefore is the number described in the last article by this property, to wit, that  $\frac{2}{3}$  of it with 4 over, will amount to the same as  $\frac{7}{2}$  of it with 9 over: and that 60 has this property will now be easily made to appear synthetically; for  $\frac{2}{3}$  of 60 is 40, and this with 4 over is 44; moreover  $\frac{7}{2}$  of 60 is 35, and this with 9 over is also 44.

*N. B.* A demonstration that proves the connection between any number and the property ascribed to it, is either analytical or synthetical: if this connection is shewn by tracing the number from the property, the demonstration of it is called an analytical demonstration; but if it is shewn by tracing the property from the number, the demonstration is then said to be synthetical.

#### Example 2.

$$\frac{2x}{3} + 12 = \frac{4x}{5} + 6.$$

Here multiply by 3, and you will have  $2x + 36 = \frac{12x}{5} + 18$ ; multiply again by 5, and you will have  $10x + 180 = 12x + 90$ ; transpose  $10x$ , and you will have  $180 = 12x - 10x + 90$ , that is,  $180 = 2x + 90$ , or rather  $2x + 90 = 180$ , for I generally choose to have the unknown quantity on the first side of the equation: transpose 90, and you will have  $2x = 180 - 90$ , that is,  $2x = 90$ ; divide by 2, and you will have  $x = 45$ .

The

## The Proof.

The original equation was  $\frac{2x}{3} + 12 = \frac{4x}{5} + 6$ : now

if  $x = 45$ , we have  $\frac{2x}{3} = 30$ , and  $\frac{2x}{3} + 12 = 42$ : again,

we have  $\frac{4x}{5} = 36$ , and  $\frac{4x}{5} + 6 = 42$ : therefore  $\frac{2x}{3} +$

$12 = \frac{4x}{5} + 6$ , because the amount of both is 42.

## Example 3.

$\frac{3x}{4} + 5 = \frac{5x}{6} + 2$ : therefore  $3x + 20 = \frac{20x}{6} + 8$ ;

therefore  $18x + 120 = 20x + 48$ ; therefore  $120 = 20x - 18x + 48$ , that is,  $120 = 2x + 48$ ; therefore  $120 - 48 = 2x$ , that is,  $2x = 72$ ; therefore  $x = 36$ .

## The Proof.

The original equation was  $\frac{3x}{4} + 5 = \frac{5x}{6} + 2$ : now if

$x = 36$ , we shall have  $\frac{3x}{4} = 27$ , and  $\frac{3x}{4} + 5 = 32$ : we

shall also have  $\frac{5x}{6} = 30$ , and  $\frac{5x}{6} + 2 = 32$ ; therefore

if  $x = 36$ , we shall have  $\frac{3x}{4} + 5 = \frac{5x}{6} + 2$ .

## Example 4.

$\frac{7x}{8} - 5 = \frac{9x}{10} - 8$ : therefore  $7x - 40 = \frac{72x}{10} - 64$ ;

therefore  $70x - 400 = 72x - 640$ ; therefore  $-400 = 72x - 70x - 640$ , that is,  $-400 = 2x - 640$ , or rather  $2x - 640 = -400$ : therefore  $2x = 640 - 400$ , that is,  $2x = 240$ ; and  $x = 120$ .

## The Proof.

The original equation,  $\frac{7x}{8} - 5 = \frac{9x}{10} - 8$ ; but  $x = 120$ ; therefore  $\frac{7x}{8} = 105$ ; therefore  $\frac{7x}{8} - 5 = 100$ ; moreover  $\frac{9x}{10} = 108$ ; therefore  $\frac{9x}{10} - 8 = 100$ ; therefore  $\frac{7x}{8} - 5 = \frac{9x}{10} - 8$ .

## Example 5.

$\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$ : therefore  $5x - 72 = 666 - \frac{63x}{12}$ ; therefore  $60x - 864 = 7992 - 63x$ ; therefore  $60x + 63x - 864 = 7992$ , that is,  $123x - 864 = 7992$ ; therefore  $123x = 7992 + 864$ , that is,  $123x = 8856$ ; and  $x = 72$ .

## The Proof.

The original equation,  $\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$ ;  $x = 72$ ; therefore  $\frac{5x}{9} = 40$ ; therefore  $\frac{5x}{9} - 8 = 32$ ; again,  $\frac{7x}{12} = 42$ ; therefore  $74 - \frac{7x}{12} = 74 - 42 = 32$ : therefore  $\frac{5x}{9} - 8 = 74 - \frac{7x}{12}$ .

## Example 6.

$\frac{x}{6} - 4 = 24 - \frac{x}{8}$ : therefore  $x - 24 = 144 - \frac{6x}{8}$ ; therefore  $8x - 192 = 1152 - 6x$ ; therefore  $8x + 6x - 192 = 1152$ , that is,  $14x - 192 = 1152$ ; therefore  $14x = 1152 + 192$ , that is,  $14x = 1344$ ; and  $x = 96$ .

The

## The Proof.

The original equation,  $\frac{x}{6} - 4 = 24 - \frac{x}{8}$ ;  $x = 96$ ;  
 $\frac{x}{6} = 16$ ;  $\frac{x}{6} - 4 = 12$ : again,  $\frac{x}{8} = 12$ ; therefore  $24 -$   
 $\frac{x}{8} = 24 - 12 = 12$ ; therefore  $\frac{x}{6} - 4 = 24 - \frac{x}{8}$ .

## Example 7.

$56 - \frac{3x}{4} = 48 - \frac{5x}{8}$ : therefore  $224 - 3x = 192 -$   
 $\frac{20x}{8}$ ; therefore  $1792 - 24x = 1536 - 20x$ ; therefore  
 $1792 = 1536 + 24x - 20x$ , that is,  $1792 = 1536 + 4x$ ;  
 therefore  $1792 - 1536 = 4x$ , that is,  $4x = 256$ ; and  
 $x = 64$ .

## The Proof.

The original equation,  $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$ ;  $x =$   
 $64$ ; therefore  $\frac{3x}{4} = 48$ ; therefore  $56 - \frac{3x}{4} = 56 -$   
 $48 = 8$ : again,  $\frac{5x}{8} = 40$ ; therefore  $48 - \frac{5x}{8} = 48 - 40$   
 $= 8$ ; therefore  $56 - \frac{3x}{4} = 48 - \frac{5x}{8}$ .

## Example 8.

$36 - \frac{4x}{9} = 8$ : therefore  $324 - 4x = 72$ ; therefore  
 $324 = 72 + 4x$ ; therefore  $324 - 72 = 4x$ , that is,  $4x$   
 $= 252$ ; and  $x = 63$ .

## The Proof.

The original equation,  $36 - \frac{4x}{9} = 8$ ;  $x = 63$ ; therefore  $\frac{4x}{9} = 28$ ; therefore  $36 - \frac{4x}{9} = 36 - 28 = 8$ .

## Example 9.

$\frac{2x}{3} = \frac{176 - 4x}{5}$ ; therefore  $2x = \frac{528 - 12x}{5}$ ; therefore  $10x = 528 - 12x$ ; therefore  $10x + 12x = 528$ ; that is,  $22x = 528$ ; and  $x = 24$ .

## The Proof.

The original equation,  $\frac{2x}{3} = \frac{176 - 4x}{5}$ ;  $x = 24$ ; therefore  $\frac{2x}{3} = 16$ ; again,  $4x = 96$ ; therefore  $176 - 4x = 176 - 96 = 80$ ; therefore  $\frac{176 - 4x}{5} = \frac{80}{5} = 16$ ; therefore  $\frac{2x}{3} = \frac{176 - 4x}{5}$ .

## Example 10.

$\frac{3x}{4} + \frac{180 - 5x}{6} = 29$ ; therefore  $3x + \frac{720 - 20x}{6} = 116$ ; therefore  $18x + 720 - 20x = 696$ , that is,  $720 - 2x = 696$ ; therefore  $720 = 2x + 696$ ; therefore  $720 - 696 = 2x$ , that is,  $2x = 24$ ; and  $x = 12$ .

The

## The Proof.

The original equation,  $\frac{3x}{4} + \frac{180-5x}{6} = 29$ ;  $x = 12$ ; therefore  $\frac{3x}{4} = 9$ ;  $5x = 60$ ; therefore  $180 - 5x = 180 - 60 = 120$ ; therefore  $\frac{180-5x}{6} = \frac{120}{6} = 20$ ; therefore  $\frac{3x}{4} + \frac{180-5x}{6} = 29$ .

## Example 11.

$$\frac{45}{2x+3} = \frac{57}{4x-5}.$$

Multiply by  $2x+3$ , and you will have  $45 = \frac{114x+171}{4x-5}$ ; multiply by  $4x-5$ , and you will have  $180x-225 = 114x+171$ ; therefore  $180x-114x-225 = 171$ , that is  $66x-225 = 171$ ; therefore  $66x = 171+225$ , that is,  $66x = 396$ ; and  $x = 6$ .

## The Proof.

The original equation,  $\frac{45}{2x+3} = \frac{57}{4x-5}$ ;  $x = 6$ ; therefore  $2x = 12$ ; therefore  $2x+3 = 15$ ; therefore  $\frac{45}{2x+3} = \frac{45}{15} = 3$ ; again,  $4x = 24$ ; therefore  $4x-5 = 19$ ; therefore  $\frac{57}{4x-5} = \frac{57}{19} = 3$ ; therefore  $\frac{45}{2x+3} = \frac{57}{4x-5}$ .

## Example 12.

$\frac{128}{3x-4} = \frac{216}{5x-6}$  : therefore  $128 = \frac{648x-864}{5x-6}$  ;  
 therefore  $640x-768=648x-864$  ; therefore  $-768$   
 $=648x-640x-864$ , that is,  $-768=8x-864$  ;  
 therefore  $+864-768=8x$ , that is,  $8x=96$  ; and  
 $x=12$ .

## The Proof.

The original equation,  $\frac{128}{3x-4} = \frac{216}{5x-6}$  ;  $x=12$  ;  
 therefore  $3x=36$  ; therefore  $3x-4=32$  ; therefore  
 $\frac{128}{3x-4} = \frac{128}{32} = 4$  : again,  $5x=60$  ; therefore  $5x-6$   
 $=54$  ; therefore  $\frac{216}{5x-6} = \frac{216}{54} = 4$  ; therefore  $\frac{128}{3x-4}$   
 $= \frac{216}{5x-6}$ .

## Example 13.

$\frac{42x}{x-2} = \frac{35x}{x-3}$  : divide both numerators by  $x$ , and  
 you will have  $\frac{42}{x-2} = \frac{35}{x-3}$  ; therefore  $42 = \frac{35x-70}{x-3}$  ;  
 therefore  $42x-126=35x-70$  ; therefore  $42x-35x$   
 $-126=-70$ , that is,  $7x-126=-70$  ; therefore  
 $7x=126-70$ , that is,  $7x=56$  ; and  $x=8$ .

## The Proof.

The original equation,  $\frac{42x}{x-2} = \frac{35x}{x-3}$  ;  $x=8$  ; there-  
 fore  $x-2=6$  ;  $42x=336$  ; therefore  $\frac{42x}{x-2} = \frac{336}{6}$   
 $=56$  :

$= 56$  : again  $x - 3 = 5$  ; and  $35x = 280$  ; therefore  $\frac{35x}{x-3} = \frac{280}{5} = 56$  ; therefore  $\frac{42x}{x-2} = \frac{35x}{x-3}$ .

## Example 14.

$\frac{xx-12}{3} = \frac{xx-4}{4}$  : therefore  $xx-12 = \frac{3xx-12}{4}$  ;  
therefore  $4xx-48 = 3xx-12$  ; therefore  $4xx-3xx-48 = -12$ , that is,  $xx-48 = -12$  ; therefore  $xx = +48-12$ , that is,  $xx = 36$  ; and  $x = 6$ .

## The Proof.

The original equation,  $\frac{xx-12}{3} = \frac{xx-4}{4}$  ;  $x = 6$  ;  
therefore  $xx = 36$  ; therefore  $xx-12 = 24$  ; therefore  $\frac{xx-12}{3} = \frac{24}{3} = 8$  : again,  $xx-4 = 32$  ; therefore  $\frac{xx-4}{4} = \frac{32}{4} = 8$  ; therefore  $\frac{xx-12}{3} = \frac{xx-4}{4}$ .

## Example 15.

$\frac{5xx}{16} - 8 = 12$  ; therefore  $5xx - 128 = 192$  ; therefore  $5xx = 192 + 128$ , that is,  $5xx = 320$  ; therefore  $xx = 64$  ; and  $x = 8$ .

## The Proof.

The original equation,  $\frac{5xx}{16} - 8 = 12$  ;  $x = 8$  ;  
therefore  $xx = 64$  ; therefore  $5xx = 320$  ; therefore  $\frac{5xx}{16} = \frac{320}{16} = 20$  ; therefore  $\frac{5xx}{16} - 8 = 20 - 8 = 12$ .

## Example

## Example 16.

$615x - 7xxx = 48x$ : divide the whole by  $x$ , and you will have  $615 - 7xx = 48$ ; therefore  $615 = 7xx + 48$ ; therefore,  $615 - 48 = 7xx$ , that is,  $7xx = 567$ ; therefore,  $xx = 81$ : and  $x = 9$ .

## The Proof.

The original equation,  $615x - 7xxx = 48x$ ;  $x = 9$ ; therefore  $xx = 81$ ; therefore  $xxx = 729$ ;  $7xxx = 5103$ ; again,  $615x = 5535$ ; therefore  $615x - 7xxx = 5535 - 5103 = 432$ : lastly,  $48x = 432$ ; therefore  $615x - 7xxx = 48x$ .

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T H E

E L E M E N T S    O F    A L G E B R A .

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B O O K    I I .

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*Preparations for the Solution of Algebraic Problems.*

*Art. 25.* **I**N solving the following problems, I shall make use of a sort of mixt Algebra, using letters only in representing unknown quantities, and numbers for such as are known. This method, as I take it, will be the best to begin with : but afterwards, when my young scholar has been sufficiently exercised in this way, I shall then introduce him into pure Algebra, which he will find much more extensive than the former, not only as it enables him analytically to find out general solutions, taking in all the particular cases that can be proposed in the problem to which the solution belongs, but also as it enables him afterwards to demonstrate the same solutions or theorems synthetically.

And because I am not yet to suppose him skilled in any of the mathematical sciences, I shall draw my problems, generally speaking, from numbers, either considered abstractedly, or else as they relate to common life.

If

If a problem be justly proposed, it ought to have as many independent conditions comprehended in it, expressly or implicitly, as there are unknown quantities to be discovered by them; and it must be the chief business of an Algebraist, to search out, sift, and distinguish these conditions one from another, before ever he enters upon the solution of his problem.

I said, that so many conditions ought to be comprehended in the problem expressly or implicitly, because it may happen, that a condition may not be expressed in a problem, and yet be implied in the nature of the thing: thus in the 44th problem, where several rods are to be set upright in a straight line, at certain intervals one from another, it is implied, though not expressed, that the number of intervals must be less than the number of rods by unity.

Sometimes a condition may be introduced into a problem, that includes two or more conditions: as when we say, four numbers are in continual proportion, we mean, not only that the first number is to the second as the second is to the third, but also, that the second number is to the third as the third is to the fourth.

Whenever a problem is proposed to be solved algebraically, the Algebraist must substitute some letter of the alphabet for the unknown quantity: and if there be more unknown quantities than one, the rest must receive their names from so many conditions of the problem: and if the problem be justly stated and examined, there will still remain a condition at last, which, translated into algebraic language, will afford him an equation, the resolution whereof will give the unknown quantity for which the substitution was made; and when this unknown quantity is once discovered, the rest will be easily discovered by their names. Suppose there are four unknown quantities in a problem; then there ought to be four conditions: now the first unknown quantity receives its name arbitrarily without any condition; therefore the other  
three

three must take up three of the conditions of the problem for their names; and the fourth condition will still be left to furnish out an equation.

The learner must here be very careful to make no positions but what are sufficiently justifiable, either from the express conditions of the problem, or from the nature of the thing; all the liberty he is allowed in cases of this nature is, that he is not obliged to draw out the conditions in the same order as they are given him in the problem, but may make use of them in such an order as he thinks will be most convenient for his purpose; provided that he does not make use of the same condition twice, except in company with others that have not been considered.

My method in the forty-four following problems will be, to put down the answer immediately after the problem, and then the solution: for, in my opinion, this way of putting down the answer first, will not only serve to illustrate the following solution, but may also serve to fix the problem more firmly in the minds of young beginners, who are but too apt to neglect it, and to substitute chimerical notions of their own, that are not to be justified, either from the conditions of the problem, or common sense.

After the learner has run over some of these problems, and has got a tolerable insight into the method of their resolution, it will be very proper for him to begin again, and to attempt the solution of every problem himself, and not to have recourse to the solutions here given, but in cases of absolute necessity: but after the work is over, he may then compare his own solution with that which is here given, and may alter or reform it as he thinks fit.

*The Solution of some Problems producing simple Equations.*

PROBLEM I.

26. *What two numbers are those, whose difference is 14, and whose sum, when added together, is 48?*

*Ans.*

*Ans.* The numbers are 31 and 17: for  $31 - 17 = 14$ ; and  $31 + 17 = 48$ .

### SOLUTION.

In this problem there are two unknown quantities, to wit, the two numbers sought; and there are two conditions; first, that the less number when subtracted from the greater must leave 14; and secondly, that the two numbers when added together must make 48: therefore I put  $x$  for the less number; and to find a name for the greater, I have recourse to the first condition of the problem, which informs me, that the difference betwixt the two numbers sought is 14; therefore, if I call the less number  $x$ , I ought to call the greater  $x + 14$ : thus then I have got names for both my unknown quantities, and have still a condition in reserve for an equation, which is the second: now according to this second condition, the two numbers sought, when added together, must make 48; therefore  $x$  and  $x + 14$ , when added together, must make 48; but  $x$  and  $x + 14$ , when added together, make  $2x + 14$ ; whence I have this equation,  $2x + 14 = 48$ ; therefore  $2x = 48 - 14 = 34$ ; therefore  $x$ , or the less number  $= 17$ , and  $x + 14$ , or the greater number  $= 31$ , as above.

In our solution of this problem, the notation was drawn from the first condition, and the equation from the second; but the notation might have been drawn from the second condition, and the equation from the first, thus: put  $x$  for the less number sought; then because the sum of both the numbers is 48, if you subtract the less number  $x$  from 48, the remainder  $48 - x$  will be the greater number, so that the two numbers sought will be  $x$ , and  $48 - x$ : subtract the former number from the latter, and the remainder or difference will be  $48 - 2x$ ; but, according to the first condition of the problem, this difference ought to be 14; therefore  $48 - 2x = 14$ : resolve this equation, and you will have  $x = 17$ , and  $48 - x = 31$ , as above.

## P R O B L E M 2.

27. *Three persons, A, B, and C, make a joint contribution, which in the whole amounts to 76 pounds: of this A contributes a certain sum unknown; B contributes as much as A, and 10 pounds more; and C as much as both A and B together: I demand their several contributions.*

*Ans.* A contributes 14 pounds, B 24, and C 38: for  $14 + 10 = 24$ , and  $14 + 24 = 38$ , and  $14 + 24 + 38 = 76$ .

## S O L U T I O N.

In this problem there are three unknown quantities, and there are three conditions for finding them out; first, that the whole contribution amounts to 76 pounds; secondly, that *B* contributes as much as *A*, and 10 pounds more; and lastly, that *C* contributes as much as both *A* and *B* together.

These things being supposed, I first put  $x$  for *A*'s contribution; then since, according to the second condition, *B* contributes as much as *A*, and 10 pounds more, I put  $x + 10$  for *B*'s contribution; lastly, since *C* contributes as much as both *A* and *B* together, I add  $x$  and  $x + 10$  together, and so put down the sum  $2x + 10$  for *C*'s contribution: thus have I got names for all my unknown quantities, and there remains still one condition unconsidered for my equation, which is, that all the contributions added together make 76 pounds; therefore I add  $x$  and  $x + 10$ , and  $2x + 10$  together, and suppose the sum  $4x + 20 = 76$ ; therefore  $4x = 76 - 20 = 56$ ; therefore  $x$ , or *A*'s contribution, equals 14;  $x + 10$ , or *B*'s contribution, equals 24; and  $2x + 10$ , or *C*'s contribution, equals 38, as above.

## P R O B L E M 3.

28. *Suppose all things as before, except that now the whole contribution amounts to 276 pounds; that of this,*

*this, A contributes a certain sum unknown; that B contributes twice as much as A, and 12 pounds more; and C three times as much as B, and 12 pounds more: I demand their several contributions.*

*Ans.* A contributes 24 pounds, B 60, and C 192: for  $24 \times 2 + 12 = 60$ ; and  $60 \times 3 + 12 = 192$ ; and  $24 + 60 + 192 = 276$ .

### SOLUTION.

Put  $x$  for *A's* contribution; then, because *B* contributes twice as much as *A*, and 12 pounds more, *B's* contribution will be  $2x + 12$ ; therefore, if *C* had contributed just three times as much as *B*, his contribution would have amounted to  $6x + 36$ ; but, according to the problem, *C* contributes this, and 12 pounds more; therefore *C's* contribution is  $6x + 48$ ; add these contributions together, to wit,  $x$ ,  $2x + 12$ , and  $6x + 48$ , and you will have  $9x + 60 = 276$ : therefore  $9x = 276 - 60 = 216$ ; and  $x$ , or *A's* contribution, equals 24; whence  $2x + 12$ , or *B's* contribution, equals 60; and  $6x + 48$ , or *C's* contribution, equals 192, as above.

### ADVERTISEMENT.

I know not whether it may not be thought impertinent here to put the learner in mind, that after  $x$  was found equal to 24, the other two unknown quantities,  $2x + 12$ , and  $6x + 48$  were found, by substituting 24 instead of  $x$ .

### PROBLEM 4.

29. *One begins the world with a certain sum of money, which he improved so well by way of traffick, that, at the year's end, he found he had doubled his first stock, except an hundred pounds laid out in common expences; and so he went on every year doubling the last year's stock, except a hundred a year expended as before; and at the end of three years, found himself just three times as rich as at first: What was his first stock?*

*Ans.* 140 pounds : for the double of this is 280, and  $280 - 100 = 180$  pounds at the end of the first year ; the double of this last is 360, and  $360 - 100 = 260$  pounds at the end of the second year ; again, the double of this is 520, and  $520 - 100 = 420$  pounds at the end of the third year ; and 420 pounds is just three times as much as 140 pounds, his first stock.

### S O L U T I O N.

Put  $x$  for his first stock, that is, let  $x$  be the number of pounds he began with ; then the double of this is  $2x$ , and therefore he will have  $2x - 100$  at the end of the first year ; the double of this is  $4x - 200$  ; therefore he will have  $4x - 200 - 100$ , that is,  $4x - 300$  at the end of the second year ; the double of this is  $8x - 600$  ; therefore he will have  $8x - 600 - 100$ , that is,  $8x - 700$  at the end of the third year ; but according to the problem, he ought to have three times his first stock, that is,  $3x$ , at the end of the third year ; therefore  $8x - 700 = 3x$  ; therefore  $8x - 3x - 700 = 0$ , that is,  $5x - 700 = 0$  ; therefore  $5x = 700$  ; and  $x$ , or his first stock, equals 140, as above.

To this problem I shall add another of a like kind for the learner to solve himself.

*One goes with a certain quantity of money about him to a tavern, where he borrows as much as he had then about him, and out of the whole spends a shilling ; with the remainder he goes to a second tavern, where he borrows as much as he had then left, and there also spends a shilling ; and so he goes on to a third, and a fourth tavern, borrowing and spending as before ; after which he had nothing left : I demand how much money he had at first about him ?*

*Ans.*  $\frac{1}{16}$  of one shilling, that is, 11 pence farthing.

### P R O B L E M 5.

30. *One has six sons, each whereof is four years older than his next younger brother ; and the eldest is three times as old as the youngest : What are their several ages ?*

I

*Ans.*

*Ans.* 10, 14, 18, 22, 26, 30: for 30, the age of the eldest, will then be just three times 10, that is, three times the age of the youngest.

### SOLUTION.

For their several ages put  $x$ ,  $x+4$ ,  $x+8$ ,  $x+12$ ,  $x+16$ ,  $x+20$ ; then, according to the problem  $x+20$ , the age of the eldest, ought to be equal to  $3x$ , that is, three times the age of the youngest; since then  $3x = x+20$ , we shall have  $3x - x = 20$ , that is,  $2x = 20$ , and  $x = 10$ , as above.

### PROBLEM 6.

31. *There is a certain fish whose head is 9 inches; the tail is as long as the head and half the back; and the back is as long as both the head and the tail together: I demand the length of the back, and of the tail?*

*Ans.* The length of the back is 36 inches, and that of the tail 27: for  $27 = 9 + \frac{36}{2}$ ; and  $36 = 9 + 27$ .

### SOLUTION.

For the length of the back put  $x$ ; then will  $x$  be equal to the length of both head and tail together, by the supposition; therefore if from  $x$ , the length of the head and tail together, you subtract 9, the length of the head, there will remain  $x-9$  for the length of the tail; but according to the problem, the tail is as long as the head and half the back; therefore  $x-9 = \frac{x}{2} + 9$ ; therefore  $2x-18 = x+18$ ; therefore  $2x-x-18=18$ , that is,  $x-18=18$ ; and  $x$ , the length of the back, equals  $18+18=36$ ; therefore  $x-9$ , the length of the tail, equals 27, as above.

### PROBLEM 7.

32. *One has a lease for 99 years; and being asked how much of it was already expired, answered, that two-thirds of the time past was equal to four-fifths of the time to come: I demand the times past, and to come?*

*Ans.*

*Ans.* The time past was 54 years; and the whole term of years was 99; therefore the time to the expiration of the lease was 45 years: now  $\frac{2}{3}$  of 54 is 36; and  $\frac{4}{5}$  of 45 is 36.

### SOLUTION.

Put  $x$  for the time past; then, since the whole term of years was 99, if  $x$  the time past be subtracted from 99 the whole time, there will remain  $99 - x$  for the time to come; but  $\frac{2}{3}$  of the time past is  $\frac{2x}{3}$ ; and  $\frac{4}{5}$  of the time to come is  $\frac{4}{5}$  of  $\frac{99 - x}{1} = \frac{396 - 4x}{5}$ ; therefore  $\frac{2x}{3} = \frac{396 - 4x}{5}$ ; therefore  $2x = \frac{1188 - 12x}{5}$ ; therefore  $10x = 1188 - 12x$ ; therefore  $10x + 12x = 1188$ , that is,  $22x = 1188$ ; and  $x$  the time past = 54 years; therefore  $99 - x$  the time to come equals 45 years.

To this problem I shall add two others of the same nature without any solution.

First, *To divide the number 84 into two such parts, that three times one part may be equal to four times the other.*

*Ans.* The parts are 48 and 36: for in the first place,  $48 + 36 = 84$ ; and in the next place, three times 48 = 144 = four times 36.

Second, *To divide the number 60 into two such parts, that a seventh part of one may be equal to an eighth part of the other.*

*Ans.* The parts are 28 and 32; for in the first place,  $28 + 32 = 60$ ; and in the next place,  $\frac{1}{7}$  of 28 equals  $4 = \frac{1}{8}$  of 32.

### PROBLEM 8.

33. *It is required to divide the number 50 into two such parts, that  $\frac{3}{4}$  of one part being added to  $\frac{5}{6}$  of the other, may make 40.*

*Ans.* The parts are 20 and 30 : for in the first place,  $20 + 30 = 50$  ; and in the next place,  $\frac{3}{4}$  of 20, which is 15, added  $\frac{5}{6}$  of 30, which is 25, makes 40.

### SOLUTION.

Put  $x$  for one part, and consequently  $50 - x$  for the other part ; then we shall have  $\frac{3}{4}$  of  $x = \frac{3x}{4}$ , and  $\frac{5}{6}$  of  $50 - x = \frac{250 - 5x}{6}$  ; but, according to the problem, these two added together ought to make 40 ; whence we have this equation,  $\frac{3x}{4} + \frac{250 - 5x}{6} = 40$  : multiply by 4, and you will have  $3x + \frac{1000 - 20x}{6} = 160$  ; multiply again by 6, and you will have  $18x + 1000 - 20x = 960$ , that is,  $1000 - 2x = 960$  ; therefore  $1000 = 2x + 960$  ; and  $1000 - 960 = 2x$ , that is,  $2x = 40$  ; and  $x$ , which is one of the parts sought, will be 20 ; whence  $50 - x$  or the other part will be 30, as above.

Other two problems of the same nature.

*First : It is required to divide the number 20 into two such parts, that three times one part being added to five times the other may make 84.*

*Ans.* The parts are 8 and 12 : for  $8 + 12 = 20$  ; and  $8 \times 3 + 12 \times 5$ , that is,  $24 + 60 = 84$ .

*Second : It is required to divide the number 100 into two such parts, that if a third part of one be subtracted from a fourth part of the other, the remainder may be 11.*

*Ans.* The parts are 24 and 76 : for first, 24 added to 76 makes 100 ; and secondly,  $\frac{1}{3}$  part of 24, which is 8, subtracted from  $\frac{1}{4}$  of 76, which is 19, leaves 11.

## PROBLEM 9.

34. *Two persons A and B engage at play; A has 72 guineas and B 52 before they begin; and after a certain number of games won and lost between them, A rises with three times as many guineas as B: I demand how many guineas A won of B?*

*Ans.* 21: for  $72 + 21 = 93$ ; and  $52 - 21 = 31$ ; and  $93 = 31 \times 3$ .

## SOLUTION.

Put  $x$  for the number of guineas  $A$  won of  $B$ , and consequently that  $B$  lost; then will  $A$ 's last sum be  $72 + x$ , and  $B$ 's last sum  $52 - x$ ; now, according to the problem,  $A$ 's last sum is three times as much as  $B$ 's last sum; that is, three times  $52 - x$ , or  $156 - 3x$ ; therefore  $72 + x = 156 - 3x$ ; therefore  $72 + x + 3x = 156$ , that is,  $72 + 4x = 156$ ; therefore  $4x = 156 - 72 = 84$ ; therefore  $x$ , the money  $A$  won of  $B$ , equals 21 guineas, as above.

## PROBLEM 10.

35. *One meeting a company of beggars, gives to each four pence, and has sixteen-pence over: but if he would have given them six-pence apiece, he would have wanted twelve pence for that purpose: I demand the number of persons?*

*Ans.* 14: for  $14 \times 4 + 16 = 72 = 14 \times 6 - 12$ .

## SOLUTION.

Put  $x$  for the number of persons; then if he gives them four pence apiece, the number of pence given will be four times as many as the number of persons, that is  $4x$ ; therefore  $4x + 16$  will express all the money he had about him; and so also will  $6x - 12$  by a like way of reasoning; therefore  $4x + 16 = 6x - 12$ ; therefore  $16 = 6x - 4x - 12 = 2x - 12$ ; therefore  $2x = 16 + 12 = 28$ ; and  $x$ , the number of persons, equal 14, as above.

## PROBLEM II.

36. *What two numbers are those, whose difference is 4, and the difference of those squares is 112?*

*Ans.* 12 and 16 : for  $16 - 12 = 4$ , and  $16 \times 16 - 12 \times 12$ , that is,  $256 - 144 = 112$ .

## SOLUTION.

The less number, $x$ .	$x + 4$
The greater, $x + 4$ .	$x + 4$
	<hr/>
	$xx + 4x + 16$
	$+ 4x$
	<hr/>

The square of the greater,	$xx + 8x + 16$
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The square of the less,	$xx$
	<hr/>

The difference of their squares,	$* \quad 8x + 16 ;$
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whence  $8x + 16 = 112$ ; therefore  $8x = 112 - 16 = 96$ ; therefore  $x$  the less number equals 12, and  $x + 4$  the greater equals 16, as above.

## PROBLEM 12.

37. *What two numbers are those, whereof the greater is three times the less, and the sum of whose squares is five times the sum of the numbers?*

*Ans.* The numbers are 6 and 2, whose sum is 8 : now  $6 = 3$  times 2 ; and  $6 \times 6 + 2 \times 2 = 40 = 5$  times 8.

## SOLUTION.

The less number,	$x.$
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The greater,	$3x.$
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Their sum,	$4x.$
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The square of the less,	$xx.$
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The square of the greater,	$9xx.$
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The sum of their squares	$10xx.$
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But, according to the problem, the sum of their squares is 5 times the sum of their numbers, that is, 5 times  $4x$  or  $20x$ ; therefore  $10xx = 20x$ ; and  $10x$

$=20$ ; and  $x$  the less number  $=2$ ; whence  $3x$  the greater equals 6, as above.

### PROBLEM 13.

38. *What two numbers are those, whereof the less is to the greater as 2 to 3, and the product of whose multiplication is 6 times the sum of the numbers?*

*Ans.* The numbers are 10 and 15, whose sum is 25: for 10 is to 15 as 2 to 3. This will be plain by putting the question thus: if 2 gives 3, what will 10 give? for the answer will be 15: these numbers will also answer the second condition of the problem; for  $10 \times 15 = 150 = 25 \times 6$ .

### SOLUTION.

Put  $x$  for the less number; then to find the greater number say, if 2 gives 3, what will  $x$  give? and the answer is  $\frac{3x}{2}$ ; therefore, if  $x$  stands for the less num-

ber, the greater number will be  $\frac{3x}{2}$ , their sum will

be  $\frac{x}{1} + \frac{3x}{2}$ , or  $\frac{2x+3x}{2}$ , or  $\frac{5x}{2}$ ; and the product of

their multiplication  $x \times \frac{3x}{2}$ , or  $\frac{3xx}{2}$ ; but, according

to the problem, the product of their multiplication ought to be six times the sum of the numbers,

that is, six times  $\frac{5x}{2}$ , or  $\frac{30x}{2}$ ; therefore  $\frac{3xx}{2} = \frac{30x}{2}$ ;

and  $3x^2 = 30x$ ; and  $3x = 30$ ; and  $x$ , the less number,

equals 10; therefore  $\frac{3x}{2}$ , the greater number, equals

15, as above.

## PROBLEM 14.

39. *Two persons A and B were talking of their money ; says A to B, Give me five shillings of your money, and I shall have just as much as you will have left : says B to A, Rather give me five shillings of your money, and I shall then have just three times as much as you will have left : How much money had each ?*

*Ans.* *A* had 15 shillings, and *B* 25: for then, if *A* borrows 5 shillings of *B*, they will have 20 shillings each ; on the other hand, if *A* lends *B* 5 shillings, then will *A* have 10 shillings left, and *B* will have 30, which is three times as much.

## SOLUTION.

Put  $x$  for *A*'s money ; then if *A* borrows five shillings of *B*, *A* will have  $x + 5$ , and *B* by the supposition will have the same left, to wit,  $x + 5$  ; but if *B*, after having lent *A* five shillings, has  $x + 5$  left, he must have had  $x + 10$  before ; therefore if  $x$  represents *A*'s money,  $x + 10$  will represent *B*'s : let us now suppose *B* to borrow 5 shillings of *A* ; then will *B* have  $x + 15$ , and *A* will have  $x - 5$  ; but, according to the problem, *B* in this case ought to have three times as much as *A* has left, that is, three times  $x - 5$ , or  $3x - 15$  ; therefore  $3x - 15 = x + 15$  ; therefore  $3x - x - 15 = 15$ , that is,  $2x - 15 = 15$  ; therefore  $2x = 15 + 15 = 30$  ; therefore  $x$ , or *A*'s money, equals 15 shillings, and  $x + 10$ , or *B*'s, = 25, as above.

## PROBLEM 15.

40. *What two numbers are those, the product of whose multiplication is 108, and whose sum is equal to twice their difference ?*

*Ans.* 18 and 6 : for the product of their multiplication is 108 ; and their sum 24 is equal to twice their difference 12.

SOLUTION.

## SOLUTION.

For the greater number I put  $x$ ; then, had their sum been 108, I should for the other number have put  $108 - x$ ; but it is not the sum of their addition, but the product of their multiplication, that is equal to 108; therefore if one number be called  $x$ , the other will be  $\frac{108}{x}$ , which I thus demonstrate: let  $y$  be the other number; then will  $x \times y$  or  $xy = 108$  by the supposition; divide both sides of the equation by  $x$ , and you will have  $y = \frac{108}{x}$ ; as was to be demonstrated. This being admitted, the difference between the greater number  $x$ , and the less  $\frac{108}{x}$ , is  $x - \frac{108}{x}$ ; and their sum is  $x + \frac{108}{x}$ : but, by the condition of the problem, this sum ought to be equal to twice the difference, that is, to twice  $x - \frac{108}{x}$ , or  $2x - \frac{216}{x}$ ; therefore  $2x - \frac{216}{x} = x + \frac{108}{x}$ ; therefore  $2xx - 216 = xx + 108$ ; therefore  $2xx - xx - 216 = 108$ , that is,  $xx - 216 = 108$ ; therefore  $xx = 108 + 216 = 324$ ; therefore  $x$  the greater number equals 18, and  $\frac{108}{x}$  the less equals 6, as above.

## PROBLEM 16.

41. *It is required to divide the number 48 into two such parts, that one part may be three times as much above 20, as the other wants of 20.*

*Ans.* The two parts are 32 and 16: for  $32 + 16 = 48$ ; moreover 32 is 12 above 20, and 16 wants 4 of 20, and 12 is three times 4.

## SOLUTION.

Put  $x$  for the less number sought; then will  $48 - x$  be the greater, and the excess of this greater above 20 will be  $28 - x$ , as is evident by subtracting 20 from  $48 - x$ : again, the excess of 20 above the less number (which is, what the less number wants of 20) is  $20 - x$ ; and according to the problem, the former excess is three times the latter, that is, three times  $20 - x$ ; or  $60 - 3x$ ; whence we have this equation,  $28 - x = 60 - 3x$ ; therefore  $28 - x + 3x = 60$ , that is,  $28 + 2x = 60$ ; therefore  $2x = 60 - 28 = 32$ ; therefore  $x$  the less part  $= 16$ , and  $48 - x$  the greater  $= 32$ , as above.

*Another Solution of the foregoing Problem.*

Put  $x$  for what the less number wants of 20; then will the less number be  $20 - x$ , the greater  $20 + 3x$ , and their sum  $40 + 2x$ ; but, by the problem, their sum is 48; therefore  $40 + 2x = 48$ ; therefore  $2x = 48 - 40 = 8$ ; therefore  $x = 4$ ; whence  $20 - x$  the less number  $= 16$ , and  $20 + 3x$  the greater  $= 32$ .

## PROBLEM 17.

42. One has three debtors, A, B, and C, whose particular debts he has forgot; but thus much he could remember from his account, that A's and B's debts together amounted to 60 pounds; A's and C's to 80 pounds; and B's and C's to 92 pounds: I demand the particulars?

*Ans.* A's debt was 24 pounds, B's 36, and C's 56: for  $24 + 36 = 60$ ,  $24 + 56 = 80$ , and  $36 + 56 = 92$ .

## SOLUTION.

Put  $x$  for A's debt; then, because A's and B's together made 60 pounds, B's debt will be  $60 - x$ ; again, because A's and C's together made 80 pounds, C's debt must be  $80 - x$ : now since, according to the problem,

problem, *B's* and *C's* debts when added together make 92 pounds, I add  $60 - x$ , and  $80 - x$  together, and suppose the sum  $140 - 2x = 92$ ; whence  $2x + 92 = 140$ ; and  $2x = 140 - 92 = 48$ ; and  $x$ , that is, *A's* debt, = 24 pounds: whence  $60 - x$ , or *B's* debt, = 36 pounds; and  $80 - x$ , or *C's*, is 56 pounds, as above.

### PROBLEM 18.

43. *One being asked how many teeth he had remaining in his head, answered, Three times as many as he had lost; and being asked how many he had lost, answered, As many as, being multiplied into  $\frac{1}{6}$  part of the number left, would give all he ever had at first: I demand how many he had lost, and how many he had left?*

*Ans.* He had lost 8, and had 24 left: for then 24, the number left, will be equal to 3 times 8, the number lost; and moreover 8, the number lost, multiplied into 4, that is, into  $\frac{1}{6}$  part of 24, the number left, will give  $32 = 24 + 8$ , all he ever had at first.

### SOLUTION.

Teeth lost,	$x$ .
left,	$3x$ .
In all,	$4x$ .

$\frac{1}{6}$  part of the number left  $\frac{3x}{6}$ , or  $\frac{x}{2}$ ; this, multiplied into the number lost, makes  $\frac{x}{2} \times x$ , or  $\frac{xx}{2}$ ; but, according to the problem, this product is equal to all he ever had at first; whence  $\frac{xx}{2} = 4x$ ; and  $xx = 8x$ ; and  $x$ , the number lost, = 8; whence  $3x$ , the number left, = 24, as above.

### PROBLEM 19.

44. *One rents 25 acres of land, at 7 pounds 12 shillings per annum; which land consists of two sorts, the better*

ter sort he rents at 8 shillings per acre, and the worse at 5: I demand the number of acres of each sort?

*Ans.* He had 9 acres of the better sort, and 16 of the worse: for 9 times 8 shillings = 72 shillings; and 16 times 5 shillings = 80 shillings; and  $72 + 80 = 152$  shillings = 7 pounds 12 shillings.

### SOLUTION.

Put  $x$  for the number of acres of the better sort; then will  $25 - x$  be the number of acres of the worse sort, because both together make 25 acres: moreover, since he paid 8 shillings an acre for the better sort, he must pay 8 times as many shillings as he had acres, that is,  $8x$ : and since he paid 5 shillings an acre for the worse sort, he must pay 5 times as many shillings as he had acres of this sort, that is,  $25 - x \times 5$ , or  $125 - 5x$ : put both these rents together, and they will amount to  $8x + 125 - 5x$ , or  $3x + 125$  shillings; but they amount to 152 shillings by the supposition; therefore  $3x + 125 = 152$ ; therefore  $3x = 152 - 125 = 27$ ; therefore  $x$ , the number of acres of the better sort, = 9, and  $25 - x$ , the number of the worse sort, = 16, as above.

### PROBLEM 20.

45. One hires a labourer into his garden for 36 days, upon the following conditions: to wit, that for every day he laboured, he was to receive two shillings and six-pence; and for every day he was absent, he was to forfeit one shilling and six-pence: now at the end of 36 days, after due deductions made for his forfeitures, he received clear 2 pounds 18 shillings: I demand how many days he laboured, and how many he was absent?

*Ans.* He laboured 28 days, and loitered 8: for 28 half-crowns amount to 3 pounds 10 shillings due to him for wages; and 8 eighteen-pences amount to 12 shillings due from him in forfeitures; and this latter sum, subtracted from the former, leaves 2 pounds 18 shillings to be received clear.

SOLUTION.

## SOLUTION.

Put  $x$  for the number of days he laboured; then will  $36 - x$  represent the number of days he was absent: again, since he was to receive 30 pence for every day he laboured, the pence due to him in wages will be  $30 \times x$ , or  $30x$ ; and since he was to forfeit 18 pence for every day he was absent, the pence due from him in forfeitures will be  $18 \times 36 - x$ , or  $648 - 18x$ : subtract now  $648 - 18x$ , the pence due from him in forfeitures, from  $30x$ , the pence due to him for wages; or, which is all one, add  $18x - 648$  to  $30x$ , and there arises  $48x - 648$ , the pence to be received clear: but he received clear 2 pounds 18 shillings, or 696 pence, by the supposition; therefore  $48x - 648 = 696$ ; therefore  $48x = 648 + 696 = 1344$ ; therefore  $x$ , the number of days he laboured,  $= 28$ ; and  $36 - x$ , the number of days he loitered,  $= 8$ , as above.

## PROBLEM 22.

47. *One lets out a certain sum of money at 6 per cent. simple interest; which interest in 10 years time wanted but 12 pounds of the principal: What was the principal?*

*Ans.* The principal was 30 pounds, and the interest 18 pounds  $= 30 - 12$ : for as 100 pounds principal is to its annual interest 6 pounds, so is 30 pounds principal to its annual interest 1.8 pounds; and therefore its 10 years interest will be 18 pounds.

## SOLUTION.

Put  $x$  for the number of pounds in the principal; then, to find its interest for one year, say, if 100 pound principal give 6 pounds interest, what will  $x$  principal give? and the answer will be  $\frac{6x}{100}$ ; this will be the interest of  $x$  for one year, and therefore its interest

interest for ten years will be  $\frac{60x}{100}$ , or  $\frac{6x}{10}$ , or  $\frac{3x}{5}$ : but, according to the problem, this interest is to be  $x - 12$ ; for it is to want just 12 pounds of the principal, by the supposition; therefore  $x - 12 = \frac{3x}{5}$ ; therefore  $5x - 60 = 3x$ ; therefore  $5x - 3x - 60 = 0$ , that is,  $2x - 60 = 0$ ; therefore  $2x = 60$ , and  $x$  the principal  $= 30$  and  $\frac{3x}{5}$  the 10 years interest  $= 18$  pounds, as above.

### PROBLEM 23.

48. One lets out 98 pounds in two different parcels; one at 5, the other at 6 per cent. simple interest; and the interest of the whole in 15 years amounted to 81 pounds: What were the two parcels?

*Ans.* The parcel at 5 per cent. was 48 pounds, and the other at 6 per cent. was 50 pounds: for in the first place,  $48 + 50 = 98$ ; and moreover, the annual interest of 48 pounds at 5 per cent. amounts to 2 pounds 8 shillings; and the annual interest of 50 pounds at 6 per cent. is 3 pounds; therefore the whole interest amounts to 5 pounds 8 shillings in one year; and consequently to 81 pounds in 15 years.

### SOLUTION.

Put  $x$  for the number of pounds in the parcel at 5 per cent. and consequently  $98 - x$  for the number of pounds in the other parcel at 6 per cent.; then, to find the annual interest of  $x$ , say, if 100 pounds principal give 5 pounds interest, what will  $x$  give? and the answer will be  $\frac{5x}{100}$ : again, for the other parcel, say, if 100 pounds principal give 6 pounds interest, what will  $98 - x$  give? and the answer will be  $\frac{588 - 6x}{100}$ : add these two interests together, to wit,

$$\frac{5x}{100}$$

$\frac{5x}{100}$  and  $\frac{588 - 6x}{100}$ , and the sum will be  $\frac{5x + 588 - 6x}{100}$ ,  
 that is,  $\frac{588 - x}{100}$ , this is the interest of the two parcels for one year; and therefore, in 15 years time, the interest must amount to  $\frac{8820 - 15x}{100}$ ; but it amounts to 81 pounds, by the supposition; therefore  $\frac{8820 - 15x}{100} = 81$ ; therefore  $8820 - 15x = 8100$ ; therefore  $8820 = 15x + 8100$ ; therefore  $15x = 8820 - 8100 = 720$ ; therefore  $x$ , the parcel at 5 per cent. = 48 pounds; and  $98 - x$ , the parcel at 6 per cent. = 50 pounds, as above.

#### PROBLEM 24.

49. *A gentleman hires a servant for a year, or 12 months, and was to allow him for his wages six pounds in money, together with a livery cloak of a certain value agreed upon: but after seven months, upon some misdemeanour of the servant, he turns him off, with the aforesaid cloak and 50 shillings in money; which was all that was due to him for that time: I demand the value of the cloak?*

*Ans.* The value of the cloak was 48 shillings: for then his whole wages for 12 months would be 168 shillings; and, by the rule of proportion, his wages for 7 months would be 98 shillings; whence subtracting 48 shillings, the value of the cloak, there would remain 50 shillings due to him in money.

#### SOLUTION.

Put  $x$  for the value of the cloak in shillings; then will his whole wages for 12 months be  $x + 120$ ; and his wages for 7 months may be found by the golden rule, saying, as 12 is to 7, so is  $x + 120$  to  $\frac{7x + 840}{12}$ ;

but,

but, according to the problem, his wages for 7 months was the cloak and 50 shillings in money, that is,  $x + 50$ ; therefore  $x + 50 = \frac{7x + 840}{12}$ ; therefore  $12x + 600 = 7x + 840$ ; therefore  $12x - 7x + 600 = 840$ , that is,  $5x + 600 = 840$ ; therefore  $5x = 840 - 600 = 240$ ; therefore  $x$ , the value of the cloak in shillings, is 48, as above.

### PROBLEM 25.

50. One distributes 20 shillings among 20 people, giving 6 pence apiece to some, and 16 pence apiece to the rest: I demand the number of persons of each denomination.

*Ans.* There were 8 persons who received 6 pence apiece; and 12 who received 16 pence apiece: for in the first place,  $8 + 12 = 20$  persons; and since 8 sixpences are equivalent to 4 shillings, and 12 sixteen-pences to 16 shillings, we shall have, in the next place,  $4 + 16 = 20$  shillings.

### SOLUTION.

Put  $x$  for the number of persons who received 6 pence apiece; then, since there were 20 persons in all,  $20 - x$  will be the number of those who received sixteen-pence apiece: the number of pence received by the former company will be  $6x$ ; and the number of pence received by the latter will be  $20 - x \times 16$ , that is,  $320 - 16x$ ; and therefore the whole number of pence received will be  $6x + 320 - 16x$ , or  $320 - 10x$ ; but, according to the problem, there was received in the whole, 20 shillings, or 240 pence; therefore  $320 - 10x = 240$ ; therefore  $10x + 240 = 320$ ; therefore  $10x = 320 - 240 = 80$ ; therefore  $x$ , the number of persons, who received six-pence apiece, is 8, and consequently  $20 - x$ , the number of the rest, is 12, as above.

PROBLEM 26.

51. *It is required to divide 24 shillings into 24 pieces, consisting only of nine-pences and thirteen-pence-half-pennies.*

*Ans.* There must be 8 nine-pences, and 16 thirteen-pence-halfpennies; for in the first place,  $8 + 16 = 24$  pieces; and since 8 nine-pences are equivalent to 6 shillings, and 16 thirteen-pence-halfpennies to 18 shillings, we have in the next place  $6 + 18 = 24$  shillings.

SOLUTION.

Put  $x$  for the number of nine-pences, and consequently  $24 - x$  for the number of thirteen-pence-half-pennies: now the number of halfpence equivalent to the former is  $18x$ , because there are 18 halfpence in every nine-pence; and the number of halfpence equivalent to the latter is  $24 - x \times 27$ , or  $648 - 27x$ , because there are 27 halfpence in every thirteen-pence-halfpenny piece: therefore the number of halfpence equivalent to the whole will be  $18x + 648 - 27x$ , that is,  $648 - 9x$ ; but, according to the problem, the whole amounts to 24 shillings, or 576 halfpence; therefore  $648 - 9x = 576$ ; therefore  $9x + 576 = 648$ ; therefore  $9x = 648 - 576 = 72$ ; therefore  $x$ , the number of nine-pences is 8; and  $24 - x$ , the number of thirteen-pence-halfpennies, is 16, as above.

PROBLEM 27.

52. *Two persons, A and B, travelling together, A with 100, and B with 48 pounds about him, met a company of robbers, who took twice as much from A as from B, and left A thrice as much as they left B: I demand how much they took from each?*

*Ans.* They took 44 pounds from B, and twice as much, that is, 88 pounds from A, so they left B 4 pounds, and A 12 pounds, which is 3 times 4.

## SOLUTION.

Taken from  $B$ ,  $x$ .  
 from  $A$ ,  $2x$ .  
 Left  $B$ ,  $48 - x$ .  
 Left  $A$ ,  $100 - 2x$ .

But, according to the problem, they left  $A$  three times as much as they left  $B$ , that is, three times  $48 - x$ , or  $144 - 3x$ ; therefore  $100 - 2x = 144 - 3x$ ; therefore  $100 - 2x + 3x = 144$ , that is,  $100 + x = 144$ ; therefore  $x$ , the sum taken from  $B$ ,  $= 144 - 100 = 44$ ; and  $2x$ , or  $88$ , is the sum taken from  $A$ , as above.

## PROBLEM 30.

55. *There are two places 154 miles distant from each other; from whence two persons set out at the same time with a design to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in 4 hours: I demand how long and how far each travelled before they met?*

*Ans.* As our travellers were supposed both to set out at the same time, and they must both meet at the same time, it follows, that each must perform his journey in the same time; I say then, that each performed his journey in 56 hours: for if in 2 hours the first travelled 3 miles, in 56 hours he must travel 84 miles, by the rule of proportion: in like manner, if in 4 hours the second travels 5 miles, in 56 hours he must travel 70 miles; and  $84 + 70 = 154$  miles, the whole distance.

## SOLUTION.

Put  $x$  for the number of hours each travelled; then, to find how many miles the first travelled, say, if in 2 hours he travelled 3 miles, how far did he travel in  $x$  hours? and the answer is  $\frac{3x}{2}$ ; then, for the

the other, say, if in 4 hours he travelled 5 miles, how far did he travel in  $x$  hours? and the answer is  $\frac{5x}{4}$ , therefore both their journies put together make  $\frac{3x}{2} + \frac{5x}{4}$ ; but they both travelled the whole distance, 154 miles: therefore  $\frac{3x}{2} + \frac{5x}{4} = 154$ ; therefore  $3x + \frac{10x}{4} = 308$ : therefore  $12x + 10x$ , that is,  $22x = 1232$ ; therefore  $x$ , the number of hours each travelled,  $= 56$ ; therefore  $\frac{3x}{2}$ , the number of miles the first travelled,  $= 84$ ; and  $\frac{5x}{4}$ , the number of miles the second travelled,  $= 70$ , as above.

### PROBLEM 31.

56. *One sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours after, another sets out from the same place, and travels the same road at the rate of 5 miles in 3 hours: I demand how long and how far the first must travel before he is overtaken by the second?*

*Ans.* The first must travel 50 hours, and consequently 70 miles; the second must travel 50 - 8, or 42 hours, and consequently also 70 miles: since then they both set out from the same place, and the second traveller has now travelled as far as the first, he must have overtaken the first.

### SOLUTION.

Put  $x$  for the number of hours the first travelled, and consequently  $x - 8$  for the number of hours wherein the second travelled: then, to find the miles

travelled by the first, say, if in 5 hours he travels 7 miles, how far will he travel in  $x$  hours? and the answer is  $\frac{7x}{5}$ ; then, for the other, say, if in 3 hours he travelled 5 miles, how far will he travel in  $x-8$  hours, and the answer is  $\frac{5x-40}{3}$ ; but as these two travellers both set out from the same place, and must come together at the same place, it follows, that they must both travel the same length of space; therefore  $\frac{5x-40}{3} = \frac{7x}{5}$ ; therefore  $5x-40 = \frac{21x}{5}$ ; therefore  $25x-200=21x$ ; therefore  $25x-21x-200=0$ , that is,  $4x-200=0$ ; therefore  $4x=200$ ; and  $x$ , the hours travelled by the first,  $=50$ ; whence  $x-8$ , the hours travelled by the second,  $=42$ ;  $\frac{7x}{5}$ , the miles travelled by the first,  $=70$ ; and  $\frac{5x-40}{3}$ , the miles travelled by the second,  $=70$ , as above.

### PROBLEM 36.

61. *A shepherd driving a flock of sheep in time of war, meets a company of soldiers who plunder him of half his flock, and half a sheep over; the same treatment he meets with from a second, a third, and a fourth company, every succeeding company plundering him of half the flock the last had left, and half a sheep over; insomuch that at last he had but 7 sheep left: I demand how many he had at first?*

*Ans.* His flock at first consisted of 127 sheep; and if the first company had only robbed him of half his flock, they would have left him  $63\frac{1}{2}$  sheep; but as they plundered him of half his flock, and half a sheep over, they left him only 63 sheep; in like manner the second company left him 31, the third 15, and the fourth 7.

N. B. Before I enter upon the solution of this problem, I must put the learner in mind of what he has been told before (Introduction, art. 13.), to wit, that a fraction may be halved two ways, either by halving the numerator, or doubling the denominator.

### SOLUTION.

Put  $x$  for the number of his first flock, then, had the first company only taken half his flock, they would have left him the other half, viz.  $\frac{x}{2}$ ; but they took half his flock and half a sheep over; therefore they left him just so much less, to wit,  $\frac{x}{2} - \frac{1}{2}$ , or  $\frac{x-1}{2}$ : again, had the second company only taken half what remained, they would have left him half, to wit,  $\frac{x-1}{4}$ ; but, by taking half a sheep more, they left him  $\frac{x-1}{4} - \frac{1}{2}$ , that is,  $\frac{2x-2-4}{8}$ , or  $\frac{2x-6}{8}$ , or  $\frac{x-3}{4}$ ; in like manner the third company left  $\frac{x-3}{8} - \frac{1}{2}$ , or  $\frac{2x-6-8}{16}$ , or  $\frac{2x-14}{16}$ , or  $\frac{x-7}{8}$ ; and the last company left him  $\frac{x-7}{16} - \frac{1}{2}$ , or  $\frac{x-15}{16}$ ; but they left him 7 sheep by the supposition; therefore  $\frac{x-15}{16} = 7$ ; and  $x-15 = 112$ ; and  $x$ , his first number  $= 127$ , as above.

### PROBLEM 37.

62. One buys a certain number of eggs, half whereof he buys in at 2 a penny, and the other half at 3 a penny;

K 3

penny; these he afterwards sold out again at the rate of 5 for two pence, and, contrary to his expectation, lost a penny by the bargain: What was the number of his eggs?

*Ans.* The number of his eggs was 60; half whereof at two a penny cost him 15 pence; and the other half at three a penny, ten pence; and the whole 25 pence: but 60 eggs sold out at 5 for two pence, would only bring him in 24 pence, as appears by the rule of proportion; therefore he lost a penny by the bargain.

### SOLUTION.

Put  $x$  for the number of eggs; then say, if 2 eggs cost one penny, what will  $\frac{x}{2}$  one half of his eggs cost? and the answer will be  $\frac{x}{4}$ ; and for the same reason the other half at three a penny will cost him  $\frac{x}{6}$ ; so that for the whole he must pay  $\frac{x}{4} + \frac{x}{6}$ , or  $\frac{5x}{12}$ : again say, if five eggs were sold for two pence, what were  $x$  eggs sold for? and the answer will be  $\frac{2x}{5}$ ; therefore  $\frac{2x}{5}$  will be the number of pence he received for his eggs; subtract this from  $\frac{5x}{12}$ , the pence he paid for them, and the remainder  $\frac{5x}{12} - \frac{2x}{5}$ , or  $\frac{x}{60}$  will be his loss; but by the supposition, he lost one penny; therefore  $\frac{x}{60} = 1$ ; and  $x$  the number of eggs will be 60, as above.

## P R O B L E M 39.

64. *It is required to divide the number 90 into two such parts, that one part may be to the other as 2 to 3.*

*Ans.* The numbers are 36 and 54: for in the first place,  $36 + 54 = 90$ ; and in the next place, if both 36 and 54 be divided by 18, the quotients will be 2 and 3; whence I infer, that 36 is to 54 as 2 to 3; for a common division by the same number cannot alter the proportion of the numbers divided; and therefore if, after this common division, the quotients be to one another as 2 to 3, the dividends must be also in the same proportion.

## S O L U T I O N.

Put  $x$  for the less part, and  $90 - x$  for the other; then will  $x$  be to  $90 - x$  as 2 to 3, by the supposition; but by art. 15, whenever there are four proportionals, the product of the extremes will be equal to the product of the middle terms; here the extremes are  $x$  and 3, whose product is  $3x$ ; and the middle terms are  $90 - x$  and 2, whose product is  $180 - 2x$ ; therefore  $3x = 180 - 2x$ ; therefore  $5x = 180$ ; and  $x$ , the less part,  $= 36$ ; and  $90 - x$ , the greater,  $= 54$ , as above.

## P R O B L E M 41.

66. *What number is that, which, being severally added to 36 and 52, will make the former sum to the latter as 3 to 4?*

*Ans.* The number is 12: for  $36 + 12$  is to  $52 + 12$ , as 48 is to 64, as  $\frac{4}{16}$  is to  $\frac{6}{16}$ , as 3 to 4.

## S O L U T I O N.

Put  $x$  for the number sought, and you will have this proportion;  $36 + x$  is to  $52 + x$  as 3 to 4. Whence by multiplying extremes and means you will have  $144 + 4x = 156 + 3x$ ; therefore  $144 + x = 156$ ; therefore  $x$ , the number sought,  $= 12$ , as above.

## PROBLEM 42.

67. *A bookbinder sells me two paper books, one containing 48 sheets for 3 shillings and 4 pence, and another containing 75 sheets for 4 shillings and 10 pence, both bound at the same price, and both of the same sort of paper : I demand what he allows himself for binding?*

*Ans.* He reckoned 8 pence for binding ; so that the price of the paper of the first book was 32 pence, and the price of the paper of the latter 50 pence : now if this answer be just, the two prices ought to bear the same proportion to one another as the two quantities of paper ; and so we shall find them: for 32 pence are to 50 pence as  $\frac{32}{2}$  are to  $\frac{50}{2}$ , that is, as 16 to 25; and 48 sheets are to 75 sheets as  $\frac{48}{3}$  are to  $\frac{75}{3}$ , that is also, as 16 to 25.

## SOLUTION.

Put  $x$  for the number of pence reckoned for binding ; then we shall have  $40 - x$  for the price of the paper in the first book, and  $58 - x$  for the price of the paper in the second book ; and  $40 - x$  will be to  $58 - x$  as 48 to 75 ; multiply extremes and means, and you will have this equation,  $2784 - 48x = 3000 - 75x$  ; therefore  $2784 + 27x = 3000$  ; therefore  $27x = 216$  ; and  $x$ , the number of pence reckoned for binding  $= 8$ , as above.

## PROBLEM 43.

68. *What number is that, which, being severally added to 15, 27, and 45, will give three numbers in continual proportion.*

*N. B.* Three numbers are said to be in continual proportion, when the first is to the second as the second is to the third.

*Ans.* The number sought is 9 : for  $15 + 9 = 24$  ; and  $27 + 9 = 36$  ; and  $45 + 9 = 54$  ; and 24 is to 36 as 36 is to 54 : for 24 is to 36 as  $\frac{24}{12}$  is to  $\frac{36}{12}$ , that is, as 2 to

2 to 3; and 36 is to 54 as  $\frac{3}{1} \frac{6}{8}$  is to  $\frac{5}{1} \frac{4}{8}$ , that is also, as 2 to 3.

# SOLUTION.

Put  $x$  for the number sought: then we shall have this proportion,  $x+15$  is to  $x+27$ , as  $x+27$  is to  $x+45$ ; where the two middle terms are  $x+27$  and  $x+27$ : multiply extremes and means, and you will have this equation,  $xx+60x+675=xx+54x+729$ ; therefore  $60x+675=54x+729$ ; therefore  $6x+675=729$ ; therefore  $6x=54$ ; and  $x$  the number sought  $=9$ , as above.

*Of the Method of resolving Problems wherein more unknown Quantities than one are concerned, and represented by different Letters.*

70. Hitherto we have used but one single letter in every problem for some one unknown quantity in it; and if there were more, the rest received their names from the conditions of the problem; but in cases of a more complicated nature, where many unknown quantities are linked and entangled in one another, this method will be found very difficult; and therefore, in such cases, the Algebraist is allowed to use as many different letters as he has unknown quantities, provided he finds out as many independent equations for discovering their values (see the 4to edition, art. 92.); for though in every equation wherein more unknown quantities than one are concerned, they hinder one another from being found out, yet if as many fundamental equations at first be given as there are unknown quantities, it will not be difficult, in many cases, from these to derive others that are more simple, till at last you come to an equation wherein but one only unknown quantity is concerned, in which case all the rest are said to be exterminated.

Whenever two or more equations are proposed, involving as many unknown quantities, these equations

tions must first be prepared by freeing them from fractions where-ever there are any, and by ordering every particular equation so, that all the unknown quantities may possess one side of the equation, and such as are known the other; or else, that all the quantities may possess one side of the equation, and a cypher the other; it will be also convenient, that in every particular equation the unknown quantities be placed in the same order.

In laying down rules for exterminating unknown quantities, I shall begin with the simplest case first, which is that of two equations, and two unknown quantities; and when I have given as many examples as shall be thought proper in this case, I shall then proceed to others where more unknown quantities are to be exterminated.

But here I must not forget to advertise the reader, that, as I am now treating of simple equations, and problems producing such equations, I shall not meddle with any cases of extermination which lead to equations of higher forms: when I come to treat of quadratic equations, I may then perhaps add something further upon this subject; but to undertake to explain all the various methods of exterminating unknown quantities would be an endless task, and a most intolerably laborious and tedious one both to the writer and the reader, whom I cannot yet suppose to be so far gone in Analytics, as to be willing to purchase this sort of knowledge at any rate.

Let then  $x$  and  $y$  be two unknown quantities to be found out by the help of the two following equations,  $4x - 5y = 2$ , and  $6x - 7y = 4$ : or the question may be stated thus: if  $4x - 5y = 2$ , and  $6x - 7y = 4$ , what are  $x$  and  $y$ ? Now as these equations want no preparation, put them down one under another; then upon a bye piece of paper multiply the first equation ( $4x - 5y = 2$ ) by 6 the coefficient of  $x$  in the second equation, and the product will give this equation,  $24x - 30y = 12$ ; again, multiply the second equation ( $6x - 7y = 4$ ) by

4, the coefficient of  $x$  in the first equation, and the product gives  $24x - 28y = 16$ ; subtract now either of these two last equations from the other, and  $x$  will be exterminated: I choose in the present case to subtract the former equation from the latter, that the coefficient of  $y$  after subtraction may be affirmative, thus;

$$24x - 28y = 16$$

$$24x - 30y = 12$$

$$* + 2y = 4.$$

From this subtraction you have the following equation,  $2y = 4$ , which put down under the two first equations to make a third; then resolve this third equation  $2y = 4$ , and you will have  $y = 2$ , which put down under the rest for a fourth equation.

Having thus found the value of  $y = 2$ , put this value instead of  $y$  in the more simple of the two first equations, suppose in the equation  $4x - 5y = 2$ , and you will have  $4x - 10 = 2$ ; whence  $4x = 12$ , and  $x = 3$ , which put down for a fifth equation, and the work is done; for  $x$  is now found equal to 3, and  $y$  equal to 2, and these numbers three and two being substituted for  $x$  and  $y$  respectively, will answer both the conditions of the question, that is, you will have  $4x - 5y = 12 - 10 = 2$ , and  $6x - 7y = 18 - 14 = 4$ .

$$1^{\text{st}} \text{ Equ. } 4x - 5y = 2.$$

$$2^{\text{d}}, \quad 6x - 7y = 4.$$

$$3^{\text{d}}, \quad * + 2y = 4.$$

$$4^{\text{th}}, \quad * \quad y = 2.$$

$$5^{\text{th}}, \quad x \quad * = 3.$$

The coefficients of  $x$ , the quantity to be exterminated in the two first equations, were 4 and 6: now, as these numbers admit of a common divisor without any remainder, namely 2, divide them both by 2, and the quotients will be 2 and 3; use now these numbers 2 and 3 instead of 4 and 6, and the operation, as well as the equation resulting from it, will become

become more simple : for the first equation multiplied by 3 instead of 6, gives  $12x - 15y = 6$ ; and the second equation multiplied by 2 instead of 4, gives  $12x - 14y = 8$ ; and the difference of these two equations is  $y = 2$ .

Another way of exterminating the unknown quantity  $x$  is as follows : find out the value of  $x$  in respect of  $y$ , in the more simple of the two first equations ; then, substituting this value instead of  $x$  in the other equation, you will have an equation, wherein  $y$  alone is concerned : thus in the foregoing example, the first equation was  $4x - 5y = 2$ , therefore  $4x = 5y + 2$ , and  $x = \frac{5y + 2}{4}$ ; substitute now this value  $\left( \frac{5y + 2}{4} \right)$  instead of  $x$  in the second equation,  $6x - 7y = 4$ , by making  $6x = \frac{30y + 12}{4}$ , and you will have this equation,  $\frac{30y + 12}{4} - 7y = 4$ ; therefore  $30y + 12 - 28y = 16$ ; therefore  $2y + 12 = 16$ ; whence  $2y = 4$ , and  $y = 2$ ; and  $x$ , or  $\frac{5y + 2}{4} = 3$ , as before.

*N. B.* 1<sup>st</sup>, What has here been said concerning the extermination of the quantity of  $x$ , may as well be applied to the other quantity  $y$ , except that its coefficients 5 and 7 will not admit of a common divisor, as did the numbers 4 and 6.

2<sup>dly</sup>, Of the two different ways of extermination here laid down, sometimes one will be found more expeditious, and sometimes the other, as will appear by the following problems.

3<sup>dly</sup>, In the case of two unknown quantities, if the value of either of them can be had in integral terms in both equations, equate the two values one to the other, and you will have the other unknown quantity, by means whereof the first will also be known; and this makes a third way of extermination, whereof there

there are so many examples in the following problems, that nothing more needs here to be said of it.

Whenever two quantities, as  $x$  and  $y$ , are multiplied together to produce a third,  $xy$ , the two multiplicands  $x$  and  $y$  are called factors, or efficient, in which case, each is said to be the other's coefficient: thus, in the quantity  $xy$ ,  $x$  is said to be the coefficient of  $y$ , and  $y$  the coefficient of  $x$ ; therefore, if in any quantity wherein  $x$  is concerned as an efficient, its coefficient be desired; divide that quantity by  $x$ , and the quotient will be the coefficient: thus if the quantity  $12x - yx$  be divided by  $x$ , the quotient is  $12 - y$ ; therefore in the quantity  $12x - yx$ , the coefficient of  $x$  is  $12 - y$ .

#### ADVERTISEMENT.

The reader must now no longer expect to have all simple equations resolved to his hand, as hitherto has been done. If, after sixteen examples of simple equations resolved, and the solution of forty-four Algebraic problems, he be still at a loss how to reduce a simple equation, it must proceed from a weakness that either admits of no cure, or deserves none.

#### PROBLEM 45.

71. *What two numbers are those, the product of whose multiplication is 144, and the quotient of the greater divided by the less is 16?*

#### SOLUTION.

Put  $x$  for the greater number, and  $y$  for the less; and the question when abstracted from words will stand thus: if  $xy = 144$ , and  $\frac{x}{y} = 16$ , what are  $x$  and  $y$ ?

The first of these equations wants no preparation, and therefore may be put down thus:

$$\text{Equ. 1st, } xy = 144.$$

The

The second equation, when prepared according to the last art. will stand thus :

$$\text{Equ. 2d, } x - 16y = 0.$$

Multiply the first equation by 1, the supposed coefficient of  $x$  in the second, and the quotient not being altered by such a multiplication, will be  $xy = 144$  ; multiply also the second equation by  $y$ , which according to the foregoing art. is the coefficient of  $x$  in the first, and you will have  $xy - 16yy = 0$  ; subtract this latter product from the former, and you will have, Equ. 3d,  $16yy = 144$  ; whence

$$\text{Equ. 4th, } y = 3.$$

Substitute now 3 instead of  $y$ , or  $3x$  instead of  $xy$  in the first equation, and you will have  $3x = 144$ , and consequently,

$$\text{Equ. 5th, } x = 48.$$

So that the numbers at last are found to be 48 and 3 ; and they will answer the conditions of the question : for  $48 \times 3 = 144$ , and  $\frac{48}{3} = 16$ .

$$\text{Equ. 1st, } xy = 144.$$

$$2d, \quad x - 16y = 0.$$

$$3d, \quad 16yy = 144.$$

$$4th, \quad y = 3.$$

$$5th, \quad x = 48.$$

*Another Solution of the foregoing Problem, from the last Article.*

Having found from the second equation that  $x = 16y$  ; put  $16y$  for  $x$ , or  $16yy$  for  $xy$  in the first equation, and you will have  $16yy = 144$  ; whence  $y$  and  $x$  may be found as before.

#### PROBLEM 46.

72. *It is required to find two numbers with the following properties, to wit, that the first with half the second may make 20 ; and moreover, that the second, with a third part of the first, may make 20.*

SOLUTION.

## SOLUTION.

Put  $x$  for the first number, and  $y$  for the second, and the fundamental equations will be  $x + \frac{y}{2} = 20$ , and  $y + \frac{x}{3} = 20$ ; which being prepared according to art. 70, will stand thus :

$$\text{Equ. 1st, } 2x + y = 40.$$

$$\text{Equ. 2d, } x + 3y = 60.$$

Subtract the first equation from twice the second, and you will have,

$$\text{Equ. 3d, } * \quad 5y = 80 : \text{ whence}$$

$$\text{Equ. 4th, } * \quad y = 16.$$

Put 16 instead of  $y$  in the first equation, and you will have  $2x + 16 = 40$ , whence

$$\text{Equ. 5th, } x = 12.$$

Therefore the numbers sought are 12 and 16, and not 16 and 12, though 16 was found first; because  $x = 12$  was put for the first number. That these numbers will answer the conditions of the question is plain: for  $12 + \frac{16}{2}$ , or  $12 + 8 = 20$ ; and  $16 + \frac{12}{3}$ , or  $16 + 4 = 20$ .

*Another Solution, from Art. 70.*

Having found from the second equation that  $x = 60 - 3y$ , put  $60 - 3y$  for  $x$ , or  $120 - 6y$  for  $2x$  in the first equation, and you will have  $120 - 6y + y = 40$ ; whence  $y = 16$ , as before.

## PROBLEM 47.

73. *One exchanges 6 French crowns and two French dollars for 45 shillings; and at another time 9 crowns and 5 dollars of the same coin for 76 shillings: I demand the distinct values of a crown and of a dollar?*

## SOLUTION.

Put  $x$  and  $y$  for the number of shillings a crown and a dollar are respectively worth, and the equations will stand thus :

Equ.

$$\text{Equ. 1st, } 6x + 2y = 45.$$

$$\text{Equ. 2d, } 9x + 5y = 76.$$

Subtract 3 times the first equation from twice the second, and you will have,

$$\text{Equ. 3d, } * \quad 4y = 17; \text{ whence}$$

$$\text{Equ. 4th, } * \quad y = 4\frac{1}{4} \text{ shillings;}$$

that is, 4 shillings and 3 pence; put now  $4\frac{1}{4}$  for  $y$ , or  $8\frac{1}{2}$  for  $2y$  in the first equation, and you will have  $6x + 8\frac{1}{2} = 45$ , and  $6x = 36\frac{1}{2}$ , and

$$\text{Equ. 5th, } x = 6\frac{1}{12};$$

that is  $6\frac{1}{12}$  shillings, or 6 shillings and a penny; therefore the value of a crown was 6 shillings and a penny, and that of a dollar 4 shillings and 3 pence; and these values will answer the conditions of the question; for, at this rate, 6 crowns will amount to 36 shillings and 6 pence, 2 dollars to 8 shillings and 6 pence, and the whole to 45 shillings; moreover, 9 crowns will amount to 54 shillings and 9 pence, 5 dollars to 21 shillings and 3 pence, and the whole sum to 76 shillings.

#### PROBLEM 48.

74. *It is required to find two such numbers, that half the first together with a third part of the second may make 32; and moreover, that a fourth part of the first, together with a fifth part of the second, may make 18.*

#### SOLUTION.

Put  $x$  and  $y$  for the two numbers, and the fundamental equations will be  $\frac{x}{2} + \frac{y}{3} = 32$ , and  $\frac{x}{4} + \frac{y}{5} = 18$ ; which equations, when duly prepared, will stand thus:

$$\text{Equ. 1st, } 3x + 2y = 192.$$

$$\text{Equ. 2d, } 5x + 4y = 360.$$

Subtract 5 times the first equation from 3 times the second, and you will have,

$$\text{Equ. 3d, } * \quad 2y = 120; \text{ whence}$$

$$\text{Equ. 4th, } * \quad y = 60;$$

whence,

whence, and from the first equation, you will have  $3x + 2y$ , or  $3x + 120 = 192$ , which gives,

$$\text{Equ. 5th, } x = 24.$$

So the numbers are 24 and 60; and they will answer the conditions of the question:  $\frac{24}{2} + \frac{60}{3}$ , that is,  $12 + 20 = 32$ ; and moreover,  $\frac{24}{4} + \frac{60}{5}$ , that is,  $6 + 12 = 18$ .

P R O B L E M 49.

75. *Two persons A and B were talking of their ages: says A to B, 7 years ago I was just three times as old as you were, and 7 years hence I shall be just twice as old as you will be: I demand their present ages.*

S O L U T I O N.

Let  $a$  and  $b$  represent the present ages of  $A$  and  $B$  respectively; then their ages 7 years ago were  $a - 7$  and  $b - 7$ , and their ages 7 years hence will be  $a + 7$  and  $b + 7$ ; whence, and from the conditions of the problem, may be derived the two following fundamental equations:

$$a + 7 = \overline{b - 7} \times 3 = 3b - 21, \text{ and}$$

$$a + 7 = \overline{b + 7} \times 2 = 2b + 14.$$

From the former of these two equations, to wit,  $a - 7 = 3b - 21$ , we have  $a = 3b - 14$ ; from the second equation, to wit,  $a + 7 = 2b + 14$ , we have  $a = 2b + 7$ ; therefore  $3b - 14 = 2b + 7$ , since both are equal to  $a$ ; whence  $b = 21$ , and  $2b + 7$ , or  $a = 49$ .

$A$  therefore was 49 years old, and  $B$  21 years old; which is true: for then, 7 years before,  $A$ 's age would be 42, and  $B$ 's 14; and 42 is three times 14: on the other hand, 7 years after,  $A$ 's age would be 56, and  $B$ 's 28; and 56 is twice 28.

P R O B L E M 50.

76. *A jockey has two horses, A and B, whose values are sought: he has also two saddles, one valued at 12 pounds, the other at 2: now if he sets the better*  
L
saddle

*saddle upon A, and the worse saddle upon B, A will then be worth twice as much as B; but on the other hand, if he sets the better saddle upon B, and the worse saddle upon A, B will then be worth three times as much as A: I demand the values of the horses.*

## SOLUTION.

Let  $a$  and  $b$  represent the prices of the two horses  $A$  and  $B$  respectively in pounds; then if the better saddle be set upon  $A$ , and the worse upon  $B$ ,  $A$  will be worth  $a + 12$ , and  $B$  will be worth  $b + 2$ , and the first fundamental equation will be  $a + 12 = b + 2 \times 2 = 2b + 4$ ; on the other hand, if the better saddle be set upon  $B$ , and the worse upon  $A$ , then  $B$  will be worth  $b + 12$ , and  $A$  will be worth  $a + 2$ , and the second fundamental equation will be  $b + 12 = a + 2 \times 3 = 3a + 6$ : in the first fundamental equation, where  $a + 12 = 2b + 4$ , we have  $a = 2b - 8$ ; substitute therefore  $2b - 8$  instead of  $a$ , or rather  $6b - 24$  instead of  $3a$ , in the second fundamental equation (which is  $3a + 6 = b + 12$ ), and you will have  $6b - 24 + 6$ , that is,  $6b - 18 = b + 12$ ; whence  $b = 6$ , and  $2b - 8$ , or  $a = 4$ :  $A$  then was valued at 4 pounds, and  $B$  at 6, and they will answer the conditions of the question, as any one may easily try.

## PROBLEM 51.

77. *There is a certain fraction, which if an unit be added to the numerator will be equal to  $\frac{1}{3}$ ; but if on the contrary an unit be added to the denominator, the fraction will then be equivalent to  $\frac{1}{4}$ : I demand the numerator and denominator of the fraction.*

## SOLUTION.

Call the fraction  $\frac{x}{y}$ , and you will have these two fundamental equations,  $\frac{x+1}{y} = \frac{1}{3}$ , and  $\frac{x}{x+1} = \frac{1}{4}$ : the former

former of these equations, when reduced, gives  $y = 3x + 3$ , and the latter gives  $y = 4x - 1$ ; therefore  $4x - 1 = 3x + 3$ , because both are equal to  $y$ ; whence  $x$ , the numerator of the fraction, is 4; and  $3x + 3$ , or  $y$ , the denominator, is 15; and the fraction itself is,  $\frac{4}{15}$ ; which if an unit be added to the numerator, will be  $\frac{5}{15}$ , or  $\frac{1}{3}$ ; but if an unit be added to the denominator, it will be  $\frac{4}{16}$ , or  $\frac{1}{4}$ .

PROBLEM 52.

78. *There is a certain fishing rod consisting of two parts, whereof the upper part is to the lower as 5 to 7; and moreover 9 times the upper part, together with 13 times the lower, is equal to 11 times the whole rod and 36 inches over: I demand the length of the two parts.*

SOLUTION.

Put  $x$  for the length of the upper part in inches, and  $y$  for the lower; then will  $x + y$  be the length of the whole rod, and since  $x$  is to  $y$  as 5 to 7, *ex hypothesi*, by multiplying extremes and means according to art. 15, you will have  $7x = 5y$  for a fundamental equation: again, as 9 times the upper part, together with 13 times the lower, is equal to 11 times the whole rod, and 36 inches over, you have  $9x + 13y = 11x + 11y + 36$  for a second fundamental equation: the latter of these two equations gives  $x = y - 18$ , and consequently  $7x = 7y - 126$ ; substitute this value instead of  $7x$ , in the first fundamental equation, where  $7x = 5y$ , and you will have  $7y - 126 = 5y$ ; whence  $y = 63$ ; and  $y - 18$ , or  $x = 45$ .

The upper part therefore was 45 inches, and the lower 63, as will appear upon trial.

PROBLEM 53.

79. *One lays out 2 shillings and sixpence in apples and pears, buying his apples at four, and his pears at five, a penny; and afterwards accommodates his neighbour with half his apples, and one third part of his pears*

*for thirteen pence, which was the price he bought them at : I demand how many he bought of each sort ?*

### SOLUTION.

Put  $x$  for the number of apples, and  $y$  for the number of pears ; then if 4 apples cost one penny,  $x$  will cost  $\frac{x}{4}$  pence ; and for the same reason  $y$  will cost  $\frac{y}{5}$  pence, and you will have  $\frac{x}{4} + \frac{y}{5} = 30$  for a first fundamental equation : again, the price of  $\frac{x}{2}$ , half of his apples will be  $\frac{x}{8}$  and the price of  $\frac{y}{3}$ , a third part of his pears, will be  $\frac{y}{15}$  ; and you will have  $\frac{x}{8} + \frac{y}{15} = 13$  for a second fundamental equation. Hence,

$$\text{Equ. 1st, } 5x + 4y = 600.$$

$$\text{Equ. 2d, } 15x + 8y = 1560.$$

Subtract the second equation from three times the first, according to art. 70, and you will have,

$$\text{Equ. 3d, } * \quad 4y = 240 ; \text{ whence}$$

$$\text{Equ. 4th, } * \quad y = 60.$$

Substitute now 60 for  $y$ , that is, 240 for  $4y$  in the first equation  $5x + 4y = 600$ , and you will have  $5x + 240 = 600$  ; whence

$$\text{Equ. 5th, } x = 72.$$

Therefore the number of apples was 72, and the number of pears 60, as will appear upon trial.

### PROBLEM 57.

83. *A certain company at a tavern found, when they came to pay their reckoning, that if they had been three more in company to the same reckoning, they might have paid one shilling apiece less than they did ; and that, had they been two fewer in company, they must have paid one shilling apiece more than they did : I demand the number of persons, and their quota ?*

SOLUTION.

## SOLUTION.

Put  $x$  for the number of persons, and  $y$  for the number of shillings every one actually paid; now if 4 persons are to pay 5 shillings apiece, the whole reckoning must be  $4 \times 5$ , or 20 shillings; therefore if  $x$  persons are to pay  $y$  shillings apiece, the whole reckoning must be  $y \times x$  or  $xy$  shillings: this being laid down, suppose them now to be three more in company; then will the number of persons be  $x + 3$ ; and to find what every particular person ought to pay in this case, the whole reckoning  $xy$  must be divided by  $x + 3$ , the number of persons, and the quotient

$\frac{xy}{x+3}$  will be every one's particular reckoning; but,

according to the problem, every one's particular reckoning in this case would have been one shilling

less than it actually was, that is,  $y - 1$ ; therefore  $\frac{xy}{x+3}$

$= y - 1$ ; in like manner the second condition of the problem furnishes this equation,  $\frac{xy}{x-2} = y + 1$ : the first

of these equations, to wit,  $\frac{xy}{x+3} = y - 1$ , being re-

duced, gives  $x = 3y - 3$ ; and the second equation, to

wit,  $\frac{xy}{x-2} = y + 1$  being reduced gives  $x = 2y + 2$ ; there-

fore  $3y - 3 = 2y + 2$ , and  $y = 5$ ; whence  $2y + 2$ , or  $x = 12$ .

So there were 12 persons in company, their reckoning 5 shillings apiece, and their whole reckoning 3 pounds, or 60 shillings; which answers the conditions of the question: for  $\frac{60}{12} = 5$ , and  $\frac{60}{12} = 5$ .

## PROBLEM 61.

88. *What two numbers are those, whose sum is twice, and the product of whose multiplication is twelve times their difference?*

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SOLUTION.

## SOLUTION.

Put  $x$  for the greater number, and  $y$  for the less; then will their difference be  $x - y$ , their sum  $x + y$ , and the product of their multiplication  $xy$  or  $yx$ ; and the equations will be  $x + y = 2x - 2y$ , and  $yx = 12x - 12y$ ; whence

$$\text{Equ. 1st, } x - 3y = 0.$$

$$\text{Equ. 2d, } 12x - yx - 12y = 0.$$

Multiply the first equation by  $12 - y$ , which, by art. 70, is the coefficient of  $x$  in the second, and the product will be  $12x - yx - 36y + 3yy = 0$ ; subtract this equation from the second, and you will have,

$$\text{Equ. 3d, } 24y - 3yy = 0; \text{ whence}$$

$$\text{Equ. 4th, } y = 8; \text{ and}$$

$$\text{Equ. 5th, } x = 24.$$

And the numbers 24 and 8 will answer the conditions.

Otherwise thus: by the first equation  $x = 3y$ , and  $4x = 12y$ ; substitute  $4x$  for  $12y$  in the second equation, and you will have  $12x - yx - 4x = 0$ ; divide by  $x$ , and you will have  $12 - y - 4 = 0$ , and  $y = 8$ , and  $x$  or  $3y = 24$ , as before.

## PROBLEM 62.

89. *What two numbers are those, whose difference, sum and product are to each other as are the numbers two, three, and five, respectively; that is, whose difference is to their sum as two to three, and whose sum is to their product as three to five?*

## SOLUTION.

Put  $x$  for the greater number, and  $y$  for the less; then will their difference be  $x - y$ , their sum  $x + y$ , and their product  $yx$ ; and we shall have these two proportions productive of two equations, 1st,  $x - y$  is to  $x + y$  as 2 to 3, whence  $3x - 3y = 2x + 2y$ ; 2d,  $x + y$  is to  $yx$  as 3 to 5, whence  $3yx = 5x + 5y$ : the resolution follows:

Equ.

Equ. 1st,  $x - 5y = 0$ .

Equ. 2d,  $3yx - 5x - 5y = 0$ .

Multiply the first equation by  $3y - 5$ , the coefficient of  $x$  in the second, and the product will be  $3yx - 5x - 15yy + 25y = 0$ ; subtract this from the second equation, and you will have,

Equ. 3d,  $15yy - 30y = 0$ ; whence

Equ. 4th,  $y = 2$ , and

Equ. 5th,  $x = 10$ .

And the numbers 10 and 2 will answer the conditions of the problem.

Otherwise thus: by the first equation  $x = 5y$ ; substitute therefore  $x$  instead of  $5y$  in the second, and you will have  $3yx - 5x - x = 0$ ; divide by  $x$ , and you will have  $3y - 5 - 1 = 0$ , and  $y = 2$ , as before.

### PROBLEM 63.

90. *It is required to find two numbers such, that if their difference be multiplied into their sum, the product will be five; but if the difference of their squares be multiplied into the sum of their squares, the product will be sixty-five.*

### SOLUTION.

Put  $x$  for the greater number, and  $y$  for the less; then will their difference be  $x - y$ , the sum  $x + y$ , and the product of their sum and difference multiplied together will be  $x^2 - y^2$ , by art. 11; then will  $x^2 - y^2 = 5$  by the supposition, and  $x^2 = 5 + yy$ ; square both sides, and you will have  $x^4 = 25 + 10y^2 + y^4$ ; again, the difference of the squares of the two numbers sought is  $x^2 - y^2$ , and the sum of their squares  $x^2 + y^2$ , and the product of these two  $x^4 - y^4$ ; therefore  $x^4 - y^4 = 65$  by the supposition, and  $x^4 = 65 + y^4$ ; but  $x^4$  was before found equal to  $25 + 10y^2 + y^4$ ; therefore  $25 + 10y^2 + y^4 = 65 + y^4$ ; whence  $y^2 = 4$ , and  $y = 2$ ; substitute now 4 for  $y^2$  in the first fundamental equation, which was  $x^2 - y^2 = 5$ , and you will have  $x^2 - 4 = 5$ , and  $x = 3$ ; therefore the numbers sought are 3 and 2, which will answer the conditions.

## PROBLEM 65.

93. *Three persons, A, B, and C, were talking of their money; says A to B and C, Give me half of your money, and I shall have d; says B to A and C, Give me a third part of your money, and I shall have d; says C to A and B, Give me a fourth part of your money, and I shall have d. How much money had each?*

N. B. The letter *d* is here supposed to supply the place of some known quantity, which is left undetermined till the calculation is over.

## SOLUTION.

Let *a*, *b*, and *c*, represent the money of *A*, *B*, and *C*, respectively, and we shall have these three fundamental equations :

$$a + \frac{b+c}{2} = d;$$

$$b + \frac{a+c}{3} = d; \text{ and}$$

$$c + \frac{a+b}{4} = d.$$

These equations, after due preparations according to art. 70, will stand thus :

$$\text{Equ. 1st,} \quad 2a + b + c = 2d.$$

$$\text{Equ. 2d,} \quad a + 3b + c = 3d.$$

$$\text{Equ. 3d,} \quad a + b + 4c = 4d.$$

Subtract the first equation from twice the second, and you will have,

$$\text{Equ. 4th,} \quad * \quad 5b + c = 4d.$$

Subtract the third equation from the second, and you will have,

$$\text{Equ. 5th,} \quad * \quad 2b - 3c = -d.$$

Subtract five times the fifth equation from twice the fourth, and you will have,

$$\text{Equ. 6th,} \quad * \quad * \quad 17c = 13d.$$

$$\text{Equ. 7th,} \quad * \quad * \quad c = \frac{13d}{17}.$$

Put

Put this value for  $c$  in the fourth equation, and you will have  $5b + c$ , that is,  $5b + \frac{13d}{17} = 4d$ ; therefore  $85b + 13d = 68d$ , therefore  $85b = 55d$ , and  $b = \frac{55d}{85} = \frac{11d}{17}$ ; therefore

$$\text{Equ. 8th, } * \quad b \quad * = \frac{11d}{17}.$$

Put now the two values of  $b$  and  $c$  already found, instead of  $b$  and  $c$  in the first equation, and you will have  $2a + b + c$ , that is,  $2a + \frac{11d + 13d}{17}$ , or  $2a + \frac{24d}{17} = 2d$ ; whence  $34a + 24d = 34d$ ; and  $34a = 10d$ , and  $a = \frac{10d}{34} = \frac{5d}{17}$ ; therefore

$$\text{Equ. 9th, } a \quad * \quad * = \frac{5d}{17}.$$

So that the numbers are at last found to be  $a = \frac{5d}{17}$ ,  $b = \frac{11d}{17}$ , and  $c = \frac{13d}{17}$ ; whence it follows, that if any number be put for  $d$ , that will admit of the number 17 for a divisor, the quantities  $a$ ,  $b$ , and  $c$ , will come out in whole numbers: as if  $d$  be made equal to 17, the quantities  $a$ ,  $b$ , and  $c$ , will be, 5, 11, and 13, respectively; and the numbers will answer the conditions of the problem: for  $5 + \frac{11 + 13}{2}$ , or  $5 + 12 = 17$ ;  $11 + \frac{5 + 13}{3}$ , or  $11 + 6 = 17$ ;  $13 + \frac{5 + 11}{4}$ , or  $13 + 4 = 17$ .

*Advertisment.* I hope the reader does not need to be told, that the numbers  $a$ ,  $b$ , and  $c$ , must always be understood to be of the same denomination with the number  $d$ ; as, if the number  $d$  signifies so many guineas, the numbers  $a$ ,  $b$ , and  $c$ , must also signify guineas; if shillings, shillings; if pence, pence;

$$\begin{aligned} \text{Equ. 1st, } 2a + b + c &= 2d. \\ 2d, \quad a + 3b + c &= 3d.. \end{aligned}$$

Equ.

$$\begin{array}{ll}
 \text{Equ.} & 3d, \quad a + b + 4c = 4d. \\
 & 4th, \quad * \quad 5b + c = 4d. \\
 & 5th, \quad * \quad 2b - 3c = -cd. \\
 & 6th, \quad * \quad * \quad 17c = 13d. \\
 & 7th, \quad * \quad * \quad c = \frac{13d}{17}. \\
 & 8th, \quad * \quad b \quad * \quad = \frac{11d}{17}, \\
 & 9th, \quad a \quad * \quad * \quad = \frac{5d}{17}.
 \end{array}$$

## A SCHOLIUM.

94. Of the foregoing equations, the first, second, and third, wherein the quantity  $a$  is concerned, may be called equations of the first rank; the fourth and fifth, wherein the quantity  $b$  is concerned, and out of which the quantity  $a$  is excluded, may be called equations of the second rank; the sixth, wherein  $c$  is concerned, and out of which both  $a$  and  $b$  are excluded, may be called an equation of the third rank; and so on, were there ever so many unknown quantities.

Whenever the equations of any particular rank are given or found, in order to derive from thence equations of an inferior rank, the Analyst is at liberty to combine these first equations by pairs, as he pleases, provided he does but observe these two things; first, that every equation of the given rank be some time or other coupled with some other equation of the same set, so as that no equation be left out of the account; secondly, that in every particular combination, one of the equations be such as was never made use of in any combination before, and the other such as hath been concerned in some combination before, excepting the first pair. It is not to be denied but that the artist may, if he pleases, vary sometimes from this last precept: but if he always observes it, it will be altogether as well.

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THE  
ELEMENTS OF ALGEBRA.

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B O O K III.

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*Of the Composition and Resolution of a Square raised from a Binomial Root.*

101. **H**ITHERTO we have been chiefly concerned in simple equations: it is now high time to apply ourselves to the resolution of quadratics; in order to which, something must be said concerning the nature of a binomial, upon which that resolution entirely depends.

Now a binomial (at least as it is here used) is a quantity consisting of two parts or members, connected together by the sign  $+$  or  $-$ , as  $x+a$ ,  $x-a$ ,  $x+\frac{b}{2}$ ,  $x-\frac{b}{2}$ ; and a square raised from a binomial root is nothing else but the square of such a quantity: thus the square of  $x+\frac{b}{2}$  is  $xx+\frac{bx}{2}+\frac{bb}{4}$ , and that of  $x-\frac{b}{2}$  is  $xx-\frac{bx}{2}+\frac{bb}{4}$ .

$x+\frac{b}{2}$

$$x + \frac{b}{2}$$

$$x + \frac{b}{2}$$

$$\begin{array}{r} x^2 + \frac{bx}{2} + \frac{bb}{4} \\ + \frac{bx}{2} \end{array}$$

$$x^2 + \frac{2bx}{2} + \frac{bb}{4}; \text{ that is, } x^2 - \frac{2bx}{2} + \frac{bb}{4}; \text{ that is,}$$

$$x^2 + bx + \frac{bb}{4}$$

$$x - \frac{b}{2}$$

$$x - \frac{b}{2}$$

$$\begin{array}{r} x^2 - \frac{bx}{2} + \frac{bb}{4} \\ - \frac{bx}{2} \end{array}$$

$$x^2 - bx + \frac{bb}{4}$$

The difference betwixt these two squares arises from the different sign of  $b$ ; and that only affects the second member; for the third member  $\frac{bb}{4}$  will be the same, whether the quantity  $b$  be affirmative or negative; therefore, if those cases be thrown into one, it will stand thus: *The square of*  $x \pm \frac{b}{2}$  *is*  $xx \pm bx + \frac{bb}{4}$ ; *to wit,*  $+bx$  *when the root is*  $x + \frac{b}{2}$ , *and*  $-bx$  *when the root is*  $x - \frac{b}{2}$ . Now of the three members that compose this square, the first  $xx$  is the square of  $x$ , the second  $\pm bx$  is the root of that square multiplied into the coefficient  $\pm b$ ; for the root of  $xx$  is  $x$ , and  $x \times \pm b = \pm bx$ ; the third and last member  $\frac{bb}{4}$ , is the square of  $\pm \frac{b}{2}$ , that is, the square of half the coefficient of the second member; whence may be deduced the two following observations.

OBSERVATION I.

Whenever we meet with a quantity consisting of two members, as  $xx \pm bx$ , whereof one, as  $xx$ , is a square, and the other  $\pm bx$  is the root of that square multiplied into some given coefficient  $\pm b$ ; whenever, I say, we meet with such a quantity, it may be considered as an imperfect square raised from a binomial root, and may easily be compleated by adding  $\frac{bb}{4}$ , that is, by adding the square of half the coefficient of  $x$  in the second term: thus  $xx + 6x$  when compleated becomes  $xx + 6x + 9$ ;  $xx - 8x$  when compleated becomes  $xx - 8x + 16$ ;  $xx + 3x$  when compleated becomes  $xx + 3x + \frac{9}{4}$ ; for here the coefficient being 3, its half will be  $\frac{3}{2}$ , and the square of this will be  $\frac{9}{4}$ : again,  $xx + \frac{2x}{3}$  when compleated becomes  $xx + \frac{2x}{3} + \frac{1}{9}$ ; for here the second term is  $\frac{2x}{3}$ , and therefore the coefficient of  $x$  is  $\frac{2}{3}$  by art. 70; but the half of  $\frac{2}{3}$  is  $\frac{1}{3}$ , and the square of this is  $\frac{1}{9}$ : again,  $xx - \frac{5x}{6}$  when compleated becomes  $xx - \frac{5x}{6} + \frac{25}{144}$ ; for here the coefficient is  $-\frac{5}{6}$ , whose half is  $-\frac{5}{12}$ , and the square of this is  $+\frac{25}{144}$ : lastly,  $xx - \frac{bx}{a}$  when compleated becomes  $xx - \frac{bx}{a} + \frac{bb}{4aa}$ ; for here the coefficient is  $-\frac{b}{a}$ , its half  $-\frac{b}{2a}$ , and the square of this is  $\frac{bb}{4aa}$ .

## OBSERVATION 2.

In the second place it may be observed, that the root of such a square when compleated, that is, the root of  $xx \pm bx + \frac{bb}{4}$  will always be  $x \pm \frac{b}{2}$ , that is, it will always be the square root of the first member, together with half the coefficient of the second: thus the square root of  $xx + 6x + 9$  will be  $x + 3$ ; that of  $xx - 8x + 16$  will be  $x - 4$ ; that of  $xx + 3x + \frac{9}{4}$  will be  $x + \frac{3}{2}$ ; that of  $xx + \frac{2x}{3} + \frac{1}{9}$  will be  $x + \frac{1}{3}$ ; that of  $xx - \frac{bx}{a} + \frac{bb}{4aa}$  will be  $x - \frac{b}{2a}$ .

*The common Form to which all Quadratic Equations ought to be reduced in order to be resolved.*

102. Since an affected quadratic equation, as we have elsewhere defined it (art. 23), is an equation consisting of three different sorts of quantities; one sort wherein the square of the unknown quantity is concerned, another sort wherein the unknown quantity is simply concerned, and a third sort wherein it is not concerned at all; it follows, that all quadratic equations whatever may be reduced to this form, viz.  $Axx = Bx + C$ ; wherein  $A$ ,  $B$ , and  $C$ , denote known integral quantities whether affirmative or negative, and  $x$  the quantity unknown, the sign  $+$  on the latter side of the equation  $Bx + C$ , signifying no more than that the two quantities  $Bx$  and  $C$  are to be added together according to the common rules of addition, whether they be both affirmative or both negative, or one affirmative and the other negative: this will easily be allowed, if it be considered, that quadratic

tic equations, like all others, may be freed from fractions after the same manner as simple equations; and when that is done, there needs no more at most, than a bare transposition of the terms to reduce them to the form above described: we shall however give some examples of the reduction of quadratic equations to this form, amongst those that follow.

*A general Theorem for resolving all Quadratic Equations.*

103. This preparation being made, let now some general quadratic equation be proposed to be resolved, with which all particular equations may afterwards be compared, and by means whereof those equations may be more readily resolved; as for example, let the general equation in the last article be proposed, to wit  $Axx = Bx + C$ ; and let it be proposed to find the value or values of  $x$  in this equation; here, transposing  $Bx$ , I have  $Axx - Bx = C$ ; and then dividing by  $A$  in order to free  $xx$ , the highest power of  $x$ , from its coefficient, I have  $xx - \frac{Bx}{A} = \frac{C}{A}$ ; this done, I consider the first side  $xx - \frac{Bx}{A}$  as an imperfect square raised from a binomial root; and accordingly I compleat that square by art. 101, to wit, by adding  $\frac{BB}{4AA}$ , that is, by adding the square of half the coefficient of the second term; but if  $\frac{BB}{4AA}$  must be added to the first side of the equation to compleat the square, it must also be added to the other side to preserve the equality; otherwise, by an unequal addition, the equation would be destroyed: this equal addition then being made, the equation will stand thus,

thus,  $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{BB}{4AA} + \frac{C}{A}$ ; but the two fractions  $\frac{BB}{4AA}$  and  $\frac{C}{A}$ , when thrown into one, give  $\frac{AB B + 4AAC}{4AAA}$ , which, dividing by  $A$ , gives  $\frac{BB + 4AC}{4AA}$ ; therefore  $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{BB + 4AC}{4AA}$ ; therefore the square root of one side will be equal to the square root of the other; but the square root of the fraction  $\frac{BB + 4AC}{4AA}$ , at least as it here stands in letters, cannot be extracted, because, though the denominator  $4AA$  be a square, yet there is no literal quantity whatever which being multiplied into itself will produce  $BB + 4AC$ ; therefore, to put this numerator into the form of a square, let us suppose  $BB + 4AC = ss$ ; and then the equation will stand thus,  $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{ss}{4AA}$ ; but the square root of  $xx - \frac{Bx}{A} + \frac{BB}{4AA}$  is  $x - \frac{B}{2A}$ , by art. 101; and the square root of  $\frac{ss}{4AA}$  is  $\pm \frac{s}{2A}$  for a reason formerly given, to wit, because  $\frac{-s}{2A}$  when multiplied into itself will produce  $\frac{+ss}{4AA}$  as well as  $\frac{+s}{2A}$ ; and therefore, by the very definition of the square root, the former quantity has as good a right to be stiled the square root of  $\frac{ss}{4AA}$  as the latter; therefore this equation will now be reduced to a simple one, and

will stand thus,  $x - \frac{B}{2A} = \pm \frac{s}{2A}$ ; therefore  $x = \frac{B \pm s}{2A}$ ,

that is,  $x = \frac{B+s}{2A}$ , and  $x = \frac{B-s}{2A}$ . Q. E. D.

Thus we see that every quadratic equation necessarily admits of two numbers or roots (as they are called) which will equally answer the condition of the equation, that is, either of which being put equal to  $x$ , will make the two sides of the equation equal one to the other; and these two roots, in all arts and sciences where quadratic equations are concerned, are of equal estimation, whether they be affirmative or negative, or one be affirmative and the other negative: as for example, in Geometry, if a line drawn from any point towards the right hand be considered as affirmative, a line drawn from the same point to the left hand ought to be considered as negative; for let  $AB$  be any line drawn from the first point  $A$  to the point  $B$  on the right hand, and then imagine the point  $B$  to move towards  $A$ ; here then it is plain that the nearer  $B$  approaches towards  $A$ , the less will be the affirmative line  $AB$ ; when the point  $B$  coincides with  $A$ , the line  $AB$  must be looked upon as nothing, and therefore, when the point  $B$  by a continuation of its motion has passed through  $A$ , so as to lie on the left hand of  $A$ , the line  $AB$  ought now to be looked upon as negative, having passed from something through nothing into negation; and yet a line of this negative kind is as true a line as any of the affirmative kind; and therefore the negative roots of quadratic equations, which exhibit negative lines, ought to be of equal estimation with the affirmative roots that exhibit affirmative lines; and the same will be the case (I say) of all other arts and sciences where quadratic equations are concerned: but in common life, where negative quantities have no place, the affirmative roots of quadratic equations are only allowed of in the resolution

lution of problems, the negative ones being for the most part excluded.

*N. B.* 1<sup>st</sup>, The root of any quantity whether in numbers or letters, that cannot be expressed, is called a furd: thus  $\sqrt{3}$  is a furd, and so also is  $\sqrt{BB+4AC}$ ; and it was for this reason, that I made  $\sqrt{BB+4AC} = s$ , or, which is all one,  $BB+4AC = ss$ .

2<sup>dly</sup>, The quantity  $C$ , and consequently  $4AC$  will sometimes be negative; in which case the quantity  $ss$ , or  $BB+4AC$  must be looked upon as the sum of the affirmative quantity  $BB$  and the negative one  $4AC$ , when added together according to the common rules of addition.

3<sup>dly</sup>, In many of the following examples, the learner must be very careful to form a right estimation of negative quantities: thus for instance, if  $x$ , that is,  $+x = -3$ , he must make  $4x$ , or  $+4 \times -3 = -12$ ; but he must make  $-4x$ , or  $-4 \times -3 = +12$ ; so likewise  $-x$ , or  $-1x$ , or  $-1 \times -3$  must be made equal to  $+3$ , &c.

*A synthetical Demonstration of the foregoing Theorem.*

104. In the last article it was demonstrated analytically, that if  $Axx$  be equal to  $Bx+C$ , then  $x$  must necessarily be equal both to  $\frac{B+s}{2A}$ , and to  $\frac{B-s}{2A}$ , supposing  $ss$  to be equal to  $BB+4AC$ . Now it may not be improbable but that the learner, especially if he has any taste or genius, may have a curiosity to see the same demonstrated again synthetically, that is, to see it demonstrated, that if  $x$  be made equal to  $\frac{B+s}{2A}$ , or  $\frac{B-s}{2A}$ , then  $Axx$  must necessarily be equal to  $Bx+C$ : it is therefore to gratify the learner in this particular, that I have added the following demonstration.

CASE

## C A S E 1st.

Let  $x = \frac{B+s}{2A}$ ; then you will have  $xx = \frac{BB+2Bs+ss}{4AA}$ ; multiply both sides by  $A$ , and you will have  $Axx$  (or one side of the general equation) equal to  $\frac{BB+2Bs+ss}{4A}$ ; for a fraction may be multiplied by dividing the denominator, as well as by multiplying the numerator: again, since  $x = \frac{B+s}{2A}$ , you will have  $Bx = \frac{BB+B s}{2A}$ ; double both the numerator and denominator of this last fraction, which will not affect the value of the fraction, and you will have  $Bx = \frac{2BB+2Bs}{4A}$ ; therefore  $Bx+C = \frac{2BB+2Bs}{4A} + \frac{C}{1} = \frac{2BB+2Bs+4AC}{4A} = \frac{BB+2Bs+BB+4AC}{4A} = \frac{BB+2Bs+ss}{4A}$ , because  $BB+4AC=ss$  by the supposition; therefore  $Axx = Bx+C$ , since each side is equal to the same quantity  $\frac{BB+2Bs+ss}{4A}$ .

## C A S E 2d.

Let now  $x = \frac{B-s}{2A}$ , and you will have  $xx = \frac{BB-2Bs+ss}{4AA}$ , and  $Axx$  (or the first side of the general equation)  $= \frac{BB-2Bs+ss}{4A}$ : again,  $Bx = \frac{BB-Bs}{2A} = \frac{2BB-2Bs}{4A}$ ;

$\frac{2BB - 2Bs}{4A}$ ; therefore  $Bx + C = \frac{2BB - 2Bs + 4AC}{4A} =$   
 $\frac{BB - 2Bs + ss}{4A}$ ; therefore  $Axx = Bx + C$ , because each  
 side is equal to the same quantity  $\frac{BB - 2Bs + ss}{4A}$ .

*Various Examples of the Resolution of affected Quadratic Equations, both with and without the general Theorem.*

### Example 1.

105. Let the equation proposed to be resolved be  $6xx = 5x - 1$ . This particular equation, as well as all those that follow, may be resolved after the same manner as the general one in art. 103: but as these resolutions are very often attended with fractions very troublesome to the young Analyst, and as these particular equations are nothing else but particular cases of the general one, it follows, that the resolution of these equations must necessarily be included in the resolution of the general one; and consequently, that these equations will be much more easily and readily resolved by referring them to the general one: however, for the satisfaction of the learner, I shall resolve some of these equations both with and without the general theorem: and first I shall resolve the equation proposed by the help of the general theorem thus; in the general equation, art. 103, we have  $Axx = Bx + C$ ; in the particular one already proposed, we have  $6xx = 5x - 1$ ; therefore  $A$  in the general equation answers to 6 in the particular one,  $B$  answers to 5, and  $C$  to  $-1$ ; therefore, if the particular equation be referred to the general one, its resolution will be as follows:  $A = 6$ ,  $B = 5$ ,  $C = -1$ ,  $BB = 25$ ,  $4AC = -24$ ; therefore  $ss$ , or  $BB + 4AC$ , will be the sum of 25 and

$-24=1$  : therefore  $s=1$ ,  $\frac{B+s}{2A} = \frac{5+1}{12} = \frac{1}{2}$ ,  $\frac{B-s}{2A}$

$= \frac{5-1}{12} = \frac{1}{3}$ ; therefore the two roots of this equation

$6xx=5x-1$  are  $\frac{1}{2}$  and  $\frac{1}{3}$ . The resolution\* of this equation in numbers, without the general theorem, is as follows: Equation,  $6xx=5x-1$ ; therefore  $6xx-5x$

$= -1$ , and  $xx - \frac{5x}{6} = -\frac{1}{6}$ ; where  $xx - \frac{5x}{6}$  may be

considered as the two first members of a square raised from a binomial root; the coefficient of the second term is  $-\frac{5}{6}$ , its half  $-\frac{5}{12}$ , and the square of this

$\frac{+25}{12 \times 12}$ , which expression I choose to make use of rather

than  $\frac{+25}{144}$  for a reason that will presently be seen;

add now  $\frac{25}{12 \times 12}$  to both sides, that is, to one side to compleat the square, and to the other to

preserve the equality, and you will have  $xx - \frac{5x}{6} +$

$\frac{25}{12 \times 12} = \frac{-1}{6} + \frac{25}{12 \times 12}$ ; here now it is certain that

the fractions  $\frac{-1}{6}$  and  $\frac{+25}{12 \times 12}$  must be reduced to the

same denomination in order to be added together into one sum; but if this be done the common way, it will be impossible to obtain the square root of that sum without a further reduction; therefore, to avoid this, I enquire what number the denominator 6 must be multiplied by to make it  $12 \times 12$  the same with the

\* This method, of compleating the square and extracting the root of each particular equation as it occurs, I doubt not, will seem easier, and prove more satisfactory to the learner, than the use of the general theorem.

other denominator, and the answer in this case, as well as in all others of this kind, will be very easy; for  $2 \times 6 = 12$ , and therefore  $12 \times 2 \times 6$ , or  $24 \times 6 = 12 \times 12$ ; therefore I multiply both the numerator and denominator of the fraction  $\frac{-1}{6}$  into 24, and so have  $\frac{-24}{12 \times 12}$ ;

and this added to the other fraction  $\frac{+25}{12 \times 12}$  gives

$\frac{+1}{12 \times 12}$ ; and now the equation will be  $xx - \frac{5x}{6} +$

$\frac{25}{12 \times 12} = \frac{1}{12 \times 12}$ ; extract the root of both sides, and

you will have  $x - \frac{5}{12} = \pm \frac{1}{12}$ , whence  $x = \frac{5 \pm 1}{12}$ ;

but  $\frac{5+1}{12} = \frac{1}{2}$ , and  $\frac{5-1}{12} = \frac{1}{3}$ ; therefore  $x = \frac{1}{2}$ , or  $\frac{1}{3}$ .

This may also be proved synthetically thus: let  $x = \frac{1}{2}$ , then you will have  $xx = \frac{1}{4}$ , and  $6xx = \frac{6}{4}$ , or  $1\frac{1}{2}$ ; again,  $5x = \frac{5}{2} = 2\frac{1}{2}$ ; therefore  $5x - 1 = 1\frac{1}{2}$ ; therefore  $6xx = 5x - 1$ , since each equals  $1\frac{1}{2}$ .

Let us now suppose  $x = \frac{1}{3}$ , and you will have  $xx = \frac{1}{9}$ , and  $6xx = \frac{6}{9}$ , or  $\frac{2}{3}$ : on the other hand you will have  $5x = \frac{5}{3}$  or  $1\frac{2}{3}$ ; therefore  $5x - 1 = \frac{2}{3}$ ; therefore  $6xx = 5x - 1$ ; these two fractions therefore will answer the condition of the equation; and there are no other numbers beside these, whether whole numbers or fractions, that will do it.

## EXAMPLE 2.

Let the equation to be resolved be  $24x - 2xx = xx + 45$ . Here transposing  $-2xx$  we have  $3xx + 45 = 24x$ , whence  $3xx = 24x - 45$ ; and thus we have reduced the equation proposed to the form of the general one in art. 102; wherefore applying that general equation to this particular one, the resolution, by art. 103, will be as follows:  $A = 3$ ,  $B = 24$ ,  $C = -45$ ,  
BB

$BB=576$ ,  $4AC=-540$ ,  $ss=576-540=36$ ,  $s=6$ ,  
 $\frac{B+s}{2A}=5$ ,  $\frac{B-s}{2A}=3$ ; therefore  $x=5$ , or  $3$ ; and this  
 will further easily appear by substituting  $5$  or  $3$  for  $x$   
 in the original equation thus;  $x=5$ ; therefore  $24x=$   
 $120$ ;  $xx=25$ ; therefore  $24x-2xx=120-50=70$ ,  
 which is one side of the equation: on the other side  
 we have  $xx+45=25+45=70$ ; therefore  $24x-2xx$   
 $=xx+45$ . Again, let  $x=3$ , then we shall have  $24x$   
 $=72$ , and  $xx=9$ , and  $24x-2xx=54$ : on the other  
 hand,  $xx+45=54$ ; therefore  $24x-2xx=xx+45$ .

*N. B.* This last equation when reduced to the form  
 of the general one in art. 102, stood thus:  $3xx=$   
 $24x-45$ : but this equation might have been re-  
 duced \* to a more simple one of the same form by di-  
 viding the whole by  $3$ , and then the equation would  
 have stood thus;  $xx=8x-15$ : in which case we  
 should have had  $A=1$ ,  $B=8$ ,  $C=-15$ ,  $BB=64$ ,  
 $4AC=-60$ ,  $ss=4$ ,  $s=2$ ,  $\frac{B+s}{2A}=5$ ,  $\frac{B-s}{2A}=3$ , as  
 before. The solution of the foregoing equation in  
 the common way is this,  $xx-8x=-15$ ; therefore  
 completing the square,  $xx-8x+16=1$ ; therefore  
 extracting the square root,  $x-4=\pm 1$ ; therefore  $x=$   
 $+4\pm 1=5$ , or  $3$ .

### EXAMPLE 3.

Let the equation to be resolved be  $72x-2xx+144$   
 $=3xx-8x+444$ . Hence by transpositions we have  
 $72x+144=5xx-8x+444$ , and  $80x+144=5xx+$   
 $444$ , and  $5xx=80x-300$ , and  $xx=16x-60$ ; which  
 equation being resolved like that in the last example,  
 gives  $x=10$ , or  $6$ ; which may also be easily seen

\* This reduction, when it can be made without introducing  
 fractions into the equation, should never be omitted.

by substituting 10 or 6 for  $x$  in the original equation\*.

### EXAMPLE 4.

Let the equation to be resolved be  $28x - xx = 115$ . Here we have  $xx + 115 = 28x$ , and  $xx = 28x - 115$ ; which equation being resolved like that in the second example, gives  $x = 23$ , or 5; the proof whereof is easy.

### EXAMPLE 5.

Let the equation to be resolved be  $\frac{120}{x} - 5 = \frac{120}{x+4}$ ; therefore  $120 - 5x = \frac{120x}{x+4}$ ; therefore  $100x - 5xx + 480 = 120x$ ; therefore  $5xx + 120x = 100x + 480$ ; therefore  $5xx = -20x + 480$ ; therefore (dividing by 5)  $xx = -4x + 96$ ; therefore in this case,  $A = 1$ ,  $B = -4$ ,  $C = 96$ ,  $BB = 16$ ,  $4AC = 384$ ,  $ss = 16 + 384 = 400$ ,  $s = 20$ ,  $\frac{B+s}{2A} = \frac{-4+20}{2} = 8$ ,  $\frac{B-s}{2A} = \frac{-4-20}{2} = -12$ ; therefore in this equation,  $x = 8$ , or  $-12$ : the proof is thus; let  $x = +8$ ; then  $\frac{120}{x} = 15$ , and  $\frac{120}{x} - 5 = 10$ : again,  $x + 4 = 12$ , and  $\frac{120}{x+4} = 10$ ; therefore  $\frac{120}{x} - 5 = \frac{120}{x+4}$ . Again, let  $x = -12$ , then  $\frac{120}{x} = -10$ ; therefore,  $\frac{120}{x} - 5 = -10 - 5 = -15$ : on the other hand,  $x + 4 = -12 + 4 = -8$ ; therefore  $\frac{120}{x+4} = \frac{120}{-8}$

\* Or in the final equation  $xx = 16x - 60$ . This remark extends to quadratic equations in general, and to those of higher powers.

$= -15$ ; therefore  $\frac{120}{x} - 5 = \frac{120}{x+4}$ . The resolution in the common way is this:  $xx = -4x + 96$ ; therefore  $xx + 4x = 96$ ; therefore  $xx + 4x + 4 = 100$ ; therefore  $x + 2 = \pm 10$ ; therefore  $x = -2 \pm 10 = +8$ , or  $-12$ .

## EXAMPLE 6.

Let the equation to be resolved be  $2xx + 3x = 65$ ; therefore  $2xx = -3x + 65$ ; therefore in this case,  $A=2$ ,  $B=-3$ ,  $C=65$ ,  $BB=9$ ,  $4AC=520$ ,  $ss=529$ ,  $s=23$ ,  $\frac{B+s}{2A} = \frac{-3+23}{4} = 5$ ,  $\frac{B-s}{2A} = \frac{-3-23}{4} = -6\frac{1}{2}$ : therefore in this equation,  $x = +5$ , or  $-6\frac{1}{2}$ : that  $x = +5$  will easily be seen; and that  $x = -6\frac{1}{2}$ : or that  $-6\frac{1}{2}$  being substituted for  $x$  will make  $2xx + 3x = 65$ , I thus demonstrate:  $x = -6\frac{1}{2} = \frac{-13}{2}$ ; therefore  $xx = \frac{+169}{4}$ : therefore  $2xx = \frac{169}{2}$ ; and  $+3x = +3 \times \frac{-13}{2} = \frac{-39}{2}$ : therefore  $2xx + 3x = \frac{169-39}{2} = \frac{130}{2} = 65$ . The resolution in numbers;  $2xx + 3x = 65$ : therefore  $xx + \frac{3x}{2} + \frac{9}{4 \times 4} = \frac{65}{2} + \frac{9}{4 \times 4} = \frac{520+9}{4 \times 4} = \frac{529}{4 \times 4}$ ; therefore  $x + \frac{3}{4} = \pm \frac{23}{4}$ ; therefore  $x = \frac{-3 \pm 23}{4} = +5$ , or  $-6\frac{1}{2}$ .

## EXAMPLE 7,

Let the equation to be resolved be  $9xx - x = 140$ ; therefore  $9xx = 1x + 140$ . Here  $A=9$ ,  $B=1$ ,  $C=140$ ,  
 $BB$

$BB=1$ ,  $4AC=5040$ ,  $ss=5041$ ,  $s=71$ ,  $\frac{B+s}{2A}=4$ ,  
 $\frac{B-s}{2A}=-3\frac{8}{9}$ ; therefore  $x=+4$ , or  $-3\frac{8}{9}$ : the lat-  
 ter case I thus demonstrate;  $x=-3\frac{8}{9}=\frac{-35}{9}$ ;  
 therefore  $xx=\frac{+1225}{81}$ ; therefore  $9xx=\frac{1225}{9}$ : again,  
 $-1x$ , that is,  $-1 \times \frac{-35}{9}=\frac{+35}{9}$ ; therefore  $9xx-x$   
 $=\frac{1225+35}{9}=\frac{1260}{9}=140$ . In numbers thus;  
 $9xx-1x=140$ ; therefore  $xx-\frac{1x}{9}=\frac{140}{9}$ ; there-  
 fore  $xx-\frac{1x}{9}+\frac{1}{18 \times 18}=\frac{140}{9}+\frac{1}{18 \times 18}=\frac{5040+1}{18 \times 18}=\frac{5041}{18 \times 18}$ ; extract the root of both sides,  
 that is, of  $xx-\frac{1x}{9}+\frac{1}{18 \times 18}$  on one side, and of  
 $\frac{5041}{18 \times 18}$  on the other, and you will have  $x-\frac{1}{18}$   
 $=\pm\frac{71}{18}$ ; whence  $x=+4$ , or  $-3\frac{8}{9}$ .

## EXAMPLE 8.

Let the equation to be resolved be  $\frac{45}{2x+3} + \frac{116}{4x+5}=7$ ; therefore  $45 + \frac{232x+348}{4x+5}=14x+21$ ; therefore  $180x+225+232x+348=56xx+154x+105$ ; that is,  $412x+573=56xx+154x+105$ ; therefore  $258x+573=56xx+105$ ; therefore  $56xx=258x+468$ ; therefore (dividing by 2) you have  $28xx=129x+234$ ; which equation, being compared with  
 the

the general one exhibited in art. 103, gives  $A=28$ ,  $B=129$ ,  $C=234$ ,  $BB=16641$ ,  $4AC=26208$ ,  $ss=42849$ ,  $s=207$ ,  $\frac{B+s}{2A}=6$ ,  $\frac{B-s}{2A}=-1\frac{11}{28}$ ; there-

fore in this equation  $x=+6$ , or  $-1\frac{11}{28}$ ; both which I thus demonstrate: first  $x=6$ ; therefore  $2x+3=15$ ;

therefore  $\frac{45}{2x+3}=3$ ; moreover,  $4x+5=29$ ; there-

fore  $\frac{116}{4x+5}=4$ ; therefore  $\frac{45}{2x+3}+\frac{116}{4x+5}=3+4$

$=7$ : secondly,  $x=-1\frac{11}{28}=-\frac{39}{28}$ ; therefore  $2x$

$=-\frac{39}{14}$ ; therefore  $2x+3=-\frac{39}{14}+\frac{3}{1}=\frac{3}{14}$ ;

therefore  $\frac{45}{2x+3}$  is the quotient of  $\frac{45}{1}$  divided by

$\frac{3}{14}$ ; but this quotient, according to the rules of frac-

tional division, is  $\frac{630}{3}=210$ ; therefore  $\frac{45}{2x+3}$

$=210$ : again,  $4x=-\frac{39}{7}$ ; therefore  $4x+5=-\frac{39}{7}+\frac{5}{1}=-\frac{4}{7}$ ;

therefore  $\frac{116}{4x+5}$  is the quotient of

$\frac{116}{1}$  divided by  $-\frac{4}{7}$ ; but this quotient is  $-\frac{812}{4}$ ,

or  $-203$ ; therefore  $\frac{116}{4x+5}=-203$ ; therefore  $\frac{45}{2x+3}$

$+\frac{116}{4x+5}=210-203=7$ .

The resolution of this equation in the common way is as follows;  $56xx-258x=468$ ; therefore

$xx-\frac{258x}{56}=\frac{468}{56}$ ; here the coefficient of the second

term

term is  $-\frac{258}{56}$ , its half  $-\frac{129}{56}$ , and the square of this  $\frac{16641}{56 \times 56}$ ; add this square to both sides, and you will have  $xx - \frac{258x}{56} + \frac{16641}{56 \times 56} = \frac{468}{56} + \frac{16641}{56 \times 56}$   
 $= \frac{26208 + 16641}{56 \times 56} = \frac{42849}{56 \times 56}$ ; extract the square root of both sides, that is, of  $xx - \frac{258x}{56} + \frac{16641}{56 \times 56}$  on one side, and of  $\frac{42849}{56 \times 56}$  on the other, and you will have  $x - \frac{129}{56} = \pm \frac{207}{56}$ ; whence  $x = +6$ , or  $-1\frac{11}{28}$ .

## EXAMPLE 9.

Let the equation be  $15x - xx = 56$ ; then this equation being resolved by the general theorem gives  $x = 8$ , or  $7$ : and in the common way it is thus resolved;  $15x - xx = 56$ ; change all the signs to make  $xx$  affirmative, and you will have  $xx - 15x = -56$ ; whence  $xx - 15x + \frac{225}{4} = -56 + \frac{225}{4} = \frac{1}{4}$ ; therefore  $x - \frac{15}{2} = \pm \frac{1}{2}$ , and  $x = 8$ , or  $7$ ; but what I chiefly intend by this example is, to shew, that in resolving a quadratic equation by the general theorem there is no necessity of making any transposition to exhibit  $xx$  affirmative when it would otherwise have been negative; as for instance, in the equation here proposed we had  $15x - xx = 56$ ; transpose  $15x$  and you will have  $-xx$ , that is  $-1xx = -15x + 56$ ; let this equation be referred to the general one in art. 102, and resolved by the general theorem in art. 103, and you

you will have  $A = -1$ ,  $B = -15$ ,  $C = 56$ ,  $BB = 225$ ,  
 $4AC = -224$ ,  $ss = 1$ ,  $s = 1$   $\frac{B+s}{2A} = \frac{-15+1}{-2} = \frac{-14}{-2} =$   
 $+7$ ,  $\frac{B-s}{2A} = \frac{-15-1}{-2} = +8$ .

*How the Learner is to proceed when the Roots of a Quadratic Equation are inexpressible.*

106. As there are but few square numbers in comparison of the rest, and as all quadratic equations are resolved by extracting the square root, it follows, that there are but few quadratic equations capable of an exact numeral solution in comparison of those that are not : but as the square root may be extracted to any degree of exactness we please, the resolution of a quadratic equation, which depends upon it, may also be performed to any degree of accuracy whatever ; as will appear by the following example.

#### EXAMPLE 10.

Let the equation be  $xx - 4x + 1 = 0$ , or  $xx = 4x - 1$ . Here  $A = 1$ ,  $B = 4$ ,  $C = -1$ ,  $BB = 16$ ,  $4AC = -4$ ,  $ss = 12$ ,  $s = \sqrt{12}$ ,  $\frac{B+s}{2A} = \frac{4+\sqrt{12}}{2}$ , and  $\frac{B-s}{2A} = \frac{4-\sqrt{12}}{2}$  ; therefore  $x = \frac{4+\sqrt{12}}{2}$ , or  $\frac{4-\sqrt{12}}{2}$  : but let us enquire in the next place, whether these two fractions are not capable of being reduced to more simple terms : first then, it is plain that  $\frac{4}{2} = 2$ , and I say further that  $\frac{\sqrt{12}}{2} = \sqrt{3}$  ; for  $12 = 3 \times 4$  ; therefore  $\sqrt{12} = \sqrt{3} \times \sqrt{4} = \sqrt{3} \times 2$  ; therefore  $\frac{\sqrt{12}}{2} = \sqrt{3}$  ; whence it follows, that  $x = 2 + \sqrt{3}$ , or

$2 - \sqrt{3}$ ; but  $\sqrt{3}$  extracted to three decimal places gives 1.732: therefore  $2 + \sqrt{3} = 3.732$ , and  $2 - \sqrt{3} = .268$ ; therefore  $x =$  (nearly)  $3.732$ , or  $.268$ , as will be further evident from the proof following: first  $x = 3.732$ ; therefore  $xx = 13.927824$ ; and  $4x = 14.928$ ; therefore  $4x - xx = 1.000176$ ; therefore  $xx - 4x = -1.000176$ ; therefore  $xx - 4x + 1 = -.000176 = 0$  very nearly; secondly, let  $x = .268$  and you will have  $xx = .071824$  and  $4x = 1.072$ , and  $4x - xx = 1.000176$ ; therefore  $xx - 4x = -1.000176$ ; therefore  $xx - 4x + 1 = -.000176 = 0$  very nearly; therefore, in both cases, the condition of the equation is answered to as many figures or cyphers as are equal to the number of decimal places to which the square root of 3 was extracted.

It may seem to some perhaps a paradox to assert, that though the two surd values of the unknown quantity found in this and the like cases are not to be expressed in numbers, yet they may be demonstrated to be just: Thus I shall demonstrate, that if either of the two values of  $x$  found in the last case, to wit,  $2 + \sqrt{3}$ , or  $2 - \sqrt{3}$ , be substituted for  $x$ , we shall have this equation  $xx - 4x + 1 = 0$ , which was the equation there proposed: in order to this, make  $\sqrt{3} = s$ ; and first, let  $x = 2 + \sqrt{3}$ , or  $2 + s$ ; and we shall have  $xx = 4 + 4s + ss$ , and  $-4x = -8 - 4s$ ; and  $xx - 4x = 4 + 4s + ss - 8 - 4s = ss - 4$ ; but if  $s = \sqrt{3}$ ,  $ss = 3$ , and  $ss - 4 = -1$ ; therefore,  $xx - 4x = -1$ , and  $xx - 4x + 1 = 0$ : secondly, let  $x = 2 - \sqrt{3}$ , or  $2 - s$ , and we shall have  $xx = 4 - 4s + ss$ , and  $-4x = -8 + 4s$ , and  $xx - 4x = ss - 4 = -1$ , as before; whence  $xx - 4x + 1 = 0$ .

*Of impossible Roots in a Quadratic Equation, and whence they arise \*.*

107. The roots of quadratic equations are not only very often inexpressible, but sometimes even impossible, as will appear by the following example.

E X A M P L E II.

Let the equation be  $xx - 4x + 6 = 0$ , or  $xx = 4x - 6$ . Here  $A = 1$ ,  $B = 4$ ,  $C = -6$ ,  $BB = 16$ ,  $4AC = -24$ ,  $ss = -8$ ,  $s = \sqrt{-8}$ ,  $\frac{B+s}{2A} = \frac{4+\sqrt{-8}}{2}$ ,  $\frac{B-s}{2A} = \frac{4-\sqrt{-8}}{2}$ ; but  $\frac{4}{2} = 2$ , and  $-8 = -2 \times +4$ ; therefore  $\sqrt{-8} = \sqrt{-2} \times \sqrt{+4} = \sqrt{-2} \times 2$ ; therefore  $\frac{\sqrt{-8}}{2} = \sqrt{-2}$ ; therefore in this equation,  $x = 2 + \sqrt{-2}$ , or  $2 - \sqrt{-2}$ ; but as no quantity whatever, either affirmative or negative, being multiplied into itself, will produce a negative, it follows that  $\sqrt{-2}$  is not only an inexpressible quantity, but also an impossible one; and consequently, that the two values of  $x$  in this equation  $2 + \sqrt{-2}$  and  $2 - \sqrt{-2}$  will both be impossible.

N. B. Though the roots of this last equation be impossible in their own natures, yet they may be abstractedly demonstrated to be just, as in the last article, by making  $s = \sqrt{-2}$ , and consequently  $ss = -2$ .

From what has been said concerning impossible roots, it appears that one root of a quadratic equation

\* At his first going over these institutes, the learner may omit the four following articles, and go on to art. 111, in which he is also to pass over observation 2d and 3d. When he arrives at the end of Book III. he will be better prepared to understand what Dr. Saunderfon hath here delivered about impossible roots.

can never be impossible alone, but that they must either be both possible or both impossible: for it appears from the resolution of the last equation, that the impossibility of the roots flows from the impossibility of the quantity  $s$ , or of the square root of  $ss$  when it is negative; now when  $s$  is possible, both the roots of the equation  $\frac{B+s}{2A}$  and  $\frac{B-s}{2A}$  will be possible; on the other hand, when  $s$  is impossible, both the roots must necessarily be impossible.

Since the possibility or impossibility of the two roots of a quadratic equation depends upon the quantity  $ss$  being affirmative or negative, it follows, that when  $ss$ , and consequently  $s$ , equals nothing, the roots will be in the limit between possible and impossible: now if  $s=0$ , we shall have  $\frac{B+s}{2A} = \frac{B}{2A}$ , and  $\frac{B-s}{2A} = \frac{B}{2A}$ ; therefore the two unequal roots of a quadratic equation grow nearer and nearer to a state of equality as they grow nearer and nearer to a state of impossibility, but do not come to be equal till they come to the limit between possibility and impossibility.

*How to find the Sum and Product of two Roots of a Quadratic Equation without resolving it: also how to generate a Quadratic Equation that shall have any two given Numbers whatever for its Roots.*

108. In a quadratic equation of this general form, to wit,  $Axx = Bx + C$ , the sum of the roots will always be  $\frac{B}{A}$ , and the product of their multiplication  $\frac{-C}{A}$ : for

the roots of such an equation were  $\frac{B+s}{2A}$  and  $\frac{B-s}{2A}$

the sum whereof is  $\frac{2B}{2A}$ , or  $\frac{B}{A}$ ; and if these two roots be multiplied together, their product will amount to

$\frac{BB - ss}{4AA}$ ; but  $ss = BB + 4AC$  as was formerly supposed, art. 103; therefore  $ss - BB = 4AC$ , and  $BB - ss = -4AC$ ; therefore  $\frac{BB - ss}{4AA}$ , or the product of the two roots, equals  $\frac{-4AC}{4AA} = \frac{-C}{A}$ .

Therefore if  $A = 1$ , that is, if the equation be  $xx = Bx + C$ , the sum of the roots will be  $B$ , and their product  $-C$ ; that is, as the equation now stands, the sum of the roots will be the coefficient of the unknown quantity on the second side of the equation, and their product what we call the absolute term, with its sign changed.

Hence we have an easy way to form a quadratic equation whose roots shall be any two given numbers whatever: as for instance, suppose I would have a quadratic equation, whose roots shall be two, the numbers 3 and 4; here it is plain that the sum of the two numbers 3 and 4 is 7, and that the product of their multiplication is 12; therefore I form an equation whereof one side is  $xx$ , and the other side is  $7x - 12$ , to wit,  $xx = 7x - 12$ ; and the roots of this equation will be the given numbers 3 and 4, as will appear from the resolution: if I intend the two roots to be 3 and  $-4$ , their sum will be  $-1$ , and the product of their multiplication  $-12$ , and the equation  $xx = -x + 12$ : if the roots are to be  $-3$  and  $+4$ , their sum will be  $+1$ , the product of their multiplication  $-12$ , and the equation  $xx = x + 12$ : lastly, if the roots are to be  $-3$  and  $-4$ , their sum will be  $-7$ , the product of their multiplication  $+12$ , and the equation  $xx = -7x - 12$ . I shall demonstrate one general case according to the resolution given in art. 103, which will be sufficient to shew the way to all the rest: let then the roots proposed be  $p$  and  $q$ , whose sum is  $p + q$ , and the product of whose multiplication is  $pq$ ; and the equation will be  $xx = \overline{p + q} \times x - pq$ ; now if this

N

equa-

equation be referred to the general one, we shall have  $A=1$ ,  $B=\overline{p+q}$ ,  $C=-pq$ ,  $BB=pp+2pq+qq$ ,  $4AC=-4pq$ ,  $ss=pp-2pq+qq$ ,  $s=p-q$ ,  $\frac{B+s}{2A}=\frac{p+q+p-q}{2}=\frac{2p}{2}=p$ ,  $\frac{B-s}{2A}=\frac{p+q-p+q}{2}=\frac{2q}{2}=q$ ; therefore the two roots of this equation are  $p$  and  $q$ . Q. E. D.

I think I ought not to omit here, that if any one has a mind to form a quadratic equation with any two given impossible roots whatever (if I may be allowed the expression), it may be done by the foregoing rule, provided that these impossible roots be in such a form as is proper for a quadratic equation: as for example, suppose I would form a quadratic equation with these two impossible roots, to wit,  $2+\sqrt{-3}$  and  $2-\sqrt{-3}$ , I put  $ss$  for  $-3$ ; for though no possible quantity multiplied into itself can produce a negative, yet an impossible one may, that being the very thing wherein the impossibility consists; making then  $ss=-3$ , I have  $s=\sqrt{-3}$ , and so the two roots of the equation will now be  $2+s$ , and  $2-s$ ; the sum of these two roots is 4, and the product of their multiplication  $4-ss$ ; but if  $ss=-3$ ,  $-ss=+3$ , and  $4-ss=4+3=7$ ; therefore the equation with these roots will be  $xx=4x-7$ : and this will be further evident by the resolution; for if  $xx=4x-7$ , that is, if  $xx-4x=-7$ , we shall have  $xx-4x+4=-3$ , and  $x-2=\pm\sqrt{-3}$ , and  $x=2+\sqrt{-3}$ , or  $2-\sqrt{-3}$ .

*How to determine the Signs of the possible Roots of a Quadratic Equation without resolving it.*

109. If all the terms of a quadratic equation be thrown on one side of the equation, so as to be made equal to nothing; and if the term wherein  $xx$ , the square of the unknown quantity is concerned, be made the first, that wherein  $x$ , the simple power is

con-

concerned, be made the second, and the absolute term, as it is called, be made the third; the number of affirmative and negative roots in such an equation may be found by the following rule, to wit, *As often as the signs are changed in passing through all the terms from the first to the last, of so many affirmative roots will the equation consist; but as often as the signs are the same, so many negative roots will be found in the equation.* This is true in all equations whatever, though at present we shall only demonstrate it in the case of a quadratic equation: but first we shall give the following explication of the rule.

## C A S E 1.

Let the equation be  $axx - bx + c = 0$ . Here there are two changes in passing through the terms from the first to the last, to wit, from  $+axx$  to  $-bx$ , and from  $-bx$  to  $+c$ ; therefore the roots of this equation are both affirmative.

## C A S E 2.

Let the equation be  $axx - bx - c = 0$ . Here from  $+axx$  to  $-bx$  is one change, and from  $-bx$  to  $-c$  is none; therefore this equation consists of an affirmative and a negative root.

## C A S E 3.

Let the equation be  $axx + bx - c = 0$ . Here in passing from  $+axx$  to  $+bx$ , there is no change of sign, but in passing from  $+bx$  to  $-c$  there is a change; therefore this equation also consists of an affirmative and a negative root.

## C A S E 4.

Lastly, let the equation be  $axx + bx + c = 0$ . Here there are no changes, and consequently the roots of this equation are both negative. All these cases I shall demonstrate in the following manner.

## C A S E I.

Let the equation be  $axx - bx + c = 0$ , or  $axx = bx - c$ .

Here the product of the two roots is  $\frac{c}{a}$  by the last article, that is, the product of the two roots is an affirmative quantity, and therefore those roots must either be both affirmative, or both negative; but they cannot be both negative, because their sum is  $\frac{+b}{a}$ , by the same article; therefore they must both be affirmative.

## C A S E 2.

Let the equation be  $axx - bx - c = 0$ , or  $axx = bx + c$ .

Here the product of the two roots is  $-\frac{c}{a}$ , and consequently those roots must be of different kinds, one affirmative, and the other negative; and because their sum,  $+\frac{b}{a}$ , is an affirmative quantity, it is an argument that the greater root is affirmative.

## C A S E 3.

Let the equation be  $axx + bx - c = 0$ , or  $axx = -bx + c$ . Here again the product of the two roots is  $\frac{+c}{a}$ , which argues one root to be affirmative and the

other negative: and because their sum  $\frac{-b}{a}$  is a negative quantity, it is an indication that of these two roots, the greater is the negative one.

## C A S E 4.

Lastly, let the equation be  $axx + bx + c = 0$ , or  $axx = -bx - c$ . Here the product of the two roots is  $+$

$+\frac{c}{a}$  an affirmative quantity; therefore the roots are either both affirmative, or both negative; but they cannot be both affirmative, because their sum  $\frac{-b}{a}$  is negative; therefore they must both be negative.

*Impossible Roots excluded out of the foregoing Rule.*

The rule here given for determining the number of affirmative and negative roots relates only to possible roots; for impossible ones cannot be said to belong to any class, either of affirmatives or negatives; nay, so capricious are they in this respect, that, in one and the same equation, the very same impossible roots shall sometimes appear under one form, and sometimes under the other: as for example, this equation  $xx+3=0$  may be filled up two ways without affecting either the equation or its roots; to wit, either thus,  $xx-0x+3=0$ , the roots of which equation according to the foregoing rule are both affirmative; or thus,  $xx+0x+3=0$ , the roots of which equation, though it be the same with the other, and differs only in form, are both negative: the reason of this absurdity is, that the two roots of the equation  $xx+3=0$  are impossible, and occasion this confusion by putting on one shape in one equation, and another shape in the other: this will further appear from the resolution; for if  $xx+3=0$ , we have  $xx=-3$ , and  $x=+\sqrt{-3}$ , or  $-\sqrt{-3}$ , which are both impossible quantities. Again, the equation  $x^3-3=0$  may be filled up various ways; as thus,  $x^3-0x^2+0x-3=0$ , in which equation, according to the foregoing rule, there are three affirmative roots; or thus,  $x^3-0x^2-0x-3=0$ , in which equation, there is but one affirmative root and two negative ones: hence an experienced analyst would immediately conclude (as is really the case) that two of the roots of the equation  $x^3-3=0$  were impossible, and that they stood for affirmative quantities

tities in the former way of putting the equation, and for negative ones in the latter. This will further appear, when we come to treat of cubic equations.

*Of Biquadratics, and other Equations, in the Form of Quadratics.*

110. Thus much for the resolution, nature, and properties of a quadratic equation : I shall only add an example or two more of other equations that sometimes put on the form of quadratics, and have done.

E X A M P L E 12.

Let the equation to be resolved be,  $\frac{1600}{xx} + xx = 116$  ; therefore  $1600 + x^4 = 116xx$  ; therefore  $x^4 = 116xx - 1600$ . This equation is, properly speaking, a biquadratic, that is, an equation wherein the fourth power of the unknown quantity is concerned : now as every possible quadratic equation has two roots, which will equally answer the condition thereof, so a cubic equation, that is, an equation that rises to the third power of the unknown quantity, may have three such roots, a biquadratic four, &c. : but the equation  $x^4 = 116xx - 1600$ , though it be a biquadratic, and admits of four roots, yet it is in the form of a quadratic, if we consider  $xx$  as the unknown quantity ; in which case  $x^4$  must be looked upon as the square of the unknown quantity, and the equation must be referred to the general one in art. 103, thus ;  $A=1$ ,  $B=116$ ,  $C=-1600$ ,  $BB=13456$ ,  $4AC=-6400$ ,  $ss=7056$ ,  $s=84$ ,  $\frac{B-s}{2A}=100$ ,  $\frac{B-s}{2A}=16$  ; therefore in this equation,  $xx=100$ , or  $16$  : now if  $xx=100$ , we shall have  $x=+$  or  $-10$  ; if  $x^2=16$ , we shall have  $x=+$  or  $-4$  ; therefore the four roots of this biquadratic equation are,  $+10$ ,  $-10$ ,  $+4$ ,  
and

and  $-4$ : but though in this equation  $x$  has four significations,  $xx$  has but two, *viz.* 100 and 16, either of which being substituted instead of  $xx$  in the original equation, will answer that equality, as may easily be tried.

*N. B.* Whenever of the four roots of a biquadratic equation any two are equal and contrary to the other two, the equation will be in form of a quadratic, and may be resolved accordingly.

### EXAMPLE 13.

Let the equation be  $\frac{576}{xx} - xx = 55$ : here we have  $576 - x^4 = 55xx$ , and  $x^4 + 55xx = 576$ , and  $x^4 = -55x^2 + 576$ ; therefore, according to the general equation in art. 103,  $A=1$ ,  $B=-55$ ,  $C=576$ ,  $BB=3025$ ,  $4AC=2304$ ,  $ss=5329$ ,  $s=73$ ,  $\frac{B+s}{2A}=9$ ,  $\frac{B-s}{2A}=-64$ ; therefore in this equation,  $xx=+9$ , or  $-64$ : if  $xx=+9$ ,  $x=+$  or  $-3$ ; if  $xx=-64$ ,  $x$  will be equal to  $+\sqrt{-64}$ , or  $-\sqrt{-64}$ , both which values are impossible: so that in this equation  $x$  has but two values,  $+$  or  $-3$ , the other two being impossible; and  $xx$  has two values, to wit,  $+9$  and  $-64$ , which are both possible, and which, being substituted instead of  $xx$  into the original equation, will answer that equality. From this example it is easy to see, that a biquadratic equation may have four roots, and never can have more; yet it may sometimes have fewer, upon the account of some of its roots becoming impossible; nay, instances might easily be given wherein all the roots of a biquadratic equation are impossible.

If any one disapproves of the resolutions here given, he may perhaps relish the following better: let the equation be  $Ax^4=Bx^2+C$ ; here putting  $z$  for  $xx$ , and consequently  $zz$  for  $x^4$ , the equation will be changed into this common quadratic,  $Azz=Bz+C$ ; which being resolved,  $z$  or  $xx$ , and consequently  $x$  it-

self, will be known: suppose the equation to be  $Ax^6 = Bx^3 + C$ ; here putting  $z$  for  $x^3$ , the equation will be changed into a quadratic, as before, to wit,  $Azz = Bz + C$ , the resolution whereof will give  $z$  for  $x^3$ , and consequently  $x$  by an extraction of the cube root: lastly, let the equation be  $Ax = B \times \sqrt{x} + C$ ; here putting  $zz$  for  $x$ , and  $z$  for  $\sqrt{x}$ , the equation will be  $Azz = Bz + C$ , as before; whence  $z$ , and consequently  $zz$  or  $x$ , will be known.

*The Solution of some Problems producing Quadratic Equations.*

PROBLEM 69.

III. *It is required to divide the number 60 into two such parts, that the product of their multiplication may amount to 864.*

SOLUTION.

Put  $x$  for one of the parts; then will the other part be  $60 - x$ , and the product of their multiplication will be  $60x - xx$ ; whence the equation will be  $60x - xx = 864$ : therefore  $xx + 864 = 60x$ , and  $xx = 60x - 864$ : this equation, compared with the general one in art. 103, gives  $A = 1$ ,  $B = 60$ ,  $C = -864$ ,

$$BB = 3600, 4AC = -3456, ss = 144, s = 12, \frac{B+s}{2A}$$

$= 36, \frac{B-s}{2A} = 24$ ; therefore the parts sought are 24 and 36; which, upon trial, will answer the conditions of the problem.

*Obfer-*

*Observations upon the foregoing Problem.*

OBSERVATION 1st.

In this problem we may clearly see the necessity of the unknown quantity's having sometimes two distinct values in one and the same equation: for here, if I put  $x$  for the greater part of 60, the less will be  $60 - x$ , and the equation will be  $60x - xx = 864$ : suppose now I put  $x$  for the less part; then the greater will be  $60 - x$ , and the equation will still be  $60x - xx = 864$ ; therefore, whether  $x$  be put for the greater or the less part, we still fall into the same equation  $60x - xx = 864$ ; whence I infer, that this equation must either give us both the parts sought, or neither; since no reason can be shewn why it should give us one part rather than the other.

OBSERVATION 2d.

Hence also we see the necessity sometimes of impossible roots, to wit, when the cases of problems to be solved by them become impossible: as for instance, if any number, as 60, be divided into two parts, the nearer the two parts approach towards an equality, the greater will be the product of their multiplication; and therefore, if the parts be equal, the product will be the greatest possible: thus if the parts be 24 and 36, the product will be 864; if they be 25 and 35, the product will be 875; if 30 and 30, the product will be 900, which will be the greatest possible. Let us now for once put an impossible case, and let it be required to divide the number 60 into two such parts that the product of their multiplication may amount to 901; here the equation will be  $60x - xx = 901$ ; which being resolved according to art. 103, gives  $x = \frac{60 + \sqrt{-4}}{2}$ , or  $\frac{60 - \sqrt{-4}}{2}$ ; but these values

of

of  $x$  may be reduced to more simple terms thus;  
 $-4 = -1 \times +4$ ; therefore  $\sqrt{-4} = \sqrt{-1} \times \sqrt{+4} =$   
 $\sqrt{-1} \times 2$ ; therefore  $\frac{\sqrt{-4}}{2} = \sqrt{-1}$ ; but  $\frac{60}{2} = 30$ ;  
 therefore the two parts sought are  $30 + \sqrt{-1}$ , and  $30 - \sqrt{-1}$ , both which are impossible upon the account  
 of the impossibility of  $\sqrt{-1}$ ; and yet these two parts  
 abstractedly considered will answer the conditions of  
 the problem; for if  $\sqrt{-1}$  be made equal to  $s$ , the  
 two parts will be  $30 + s$  and  $30 - s$  whose sum is 60,  
 and the product of whose multiplication is  $900 - ss$ ;  
 but if  $s = \sqrt{-1}$ , we shall have  $ss = -1$ , and  $-ss =$   
 $+1$ , and  $900 - ss = 901$ ; therefore the product of  
 the two parts,  $30 + \sqrt{-1}$ , and  $30 - \sqrt{-1}$ , amount to  
 901, as was required.

## OBSERVATION 3d.

Lastly, we here also see the necessity of both the  
 roots of a quadratic equation becoming impossible at  
 once. Two impossible quantities added together  
 may sometimes make a possible one, because one  
 quantity may be as much impossible one way as the  
 other is the contrary way: thus the two impossible  
 quantities  $30 + \sqrt{-1}$  and  $30 - \sqrt{-1}$  being added to-  
 gether make 60, the impossible furds  $+\sqrt{-1}$  and  
 $-\sqrt{-1}$  destroying one another; but a possible and  
 an impossible quantity when added together can never  
 make a possible one; and therefore the two parts of  
 60 in this problem must either be both possible, or  
 both impossible.

## PROBLEM 70.

112. *There are three numbers in continual proportion, whereof the middle term is sixty, and the sum of the extremes one hundred twenty-five: What are the extremes?*

SOLUTION.

## SOLUTION.

For the extremes put  $x$  and  $125 - x$ , and you will have this proportion;  $x$  is to 60 as 60 is to  $125 - x$ , whence, by multiplying extremes and means, you have this equation,  $125x - xx = 3600$ , or  $xx + 3600 = 125x$ , or  $x^2 = 125x - 3600$ : here then  $A = 1$ ,  $B = 125$ ,  $C = -3600$ ,  $BB = 15625$ ,  $4AC = -14400$ ,  $ss = 1225$ ,  $s = 35$ ,  $\frac{B+s}{2A} = 80$ ,  $\frac{B-s}{2A} = 45$ , therefore in this equation,  $x = 45$ , or 80; but  $x$  represents either extreme, because, which extreme soever  $x$  is put for, the other will be  $125 - x$ , and the same equation will arise, to wit,  $125x - xx = 3600$ ; therefore the two extremes are 45 and 80; and they will answer the conditions of the problem; for 45 is to 60 as  $\frac{45}{60}$  is to  $\frac{60}{80}$ , that is, as 3 to 4; and 60 is to 80 as  $\frac{60}{80}$  is to  $\frac{80}{45}$ , which is also as 3 to 4.

## PROBLEM 71.

113. *It is required, having given the sum, or the difference of two numbers, together with the sum of their squares, to find the numbers.*

## SOLUTION.

*Case 1st.* Let the sum of the numbers sought be 28, and the sum of their squares 400; then putting  $x$  and  $28 - x$  for the two numbers sought, the square of the former will be  $xx$ , the square of the latter  $784 - 56x + xx$ , and the sum of their squares  $2xx - 56x + 784 = 400$ ; and the same equation will arise, whether  $x$  be made to stand for one number or the other; therefore the two values of  $x$  in this equation will be the two numbers sought; but if  $2xx - 56x + 784 = 400$ , we shall have  $2xx - 56x = -384$ ; divide the whole by 2 for a more simple equation, and you will have  $xx - 28x = -192$ ; and  $xx = 28x - 192$ ;  
which

which equation being resolved according to art. 103, gives  $x = 12$ , or  $16$ ; therefore  $12$  and  $16$  are the two numbers sought.

*Case 2d.* Let now the difference of two numbers be given, suppose  $4$ , and let the sum of their squares be  $400$ , as before; then, putting  $x$  for the less number, and  $x + 4$  for the greater, the sum of their squares will be  $2xx + 8x + 16 = 400$ ; whence  $2xx + 8x = 384$ ,  $xx + 4x = 192$ ,  $xx + 4x + 4 = 196$ ,  $x + 2 = \pm 14$ ,  $x = +12$  or  $-16$ ; now it cannot be supposed that  $+12$  and  $-16$  are the two numbers required in the problem, for their difference is  $30$ , not  $4$ ; neither ought it to be expected; for when  $x$  was put for the less number, and  $x + 4$  for the greater, the equation was  $2xx + 8x + 16 = 400$ ; but if  $x$  be put for the greater number, and consequently  $x - 4$  for the less, the equation will be  $2xx - 8x + 16 = 400$ , different from the former; since then a different equation arises according as  $x$  is put for the greater or less number, it cannot be expected that one and the same equation should give both: the true state of the case is this; there are two pairs of numbers which will equally solve this question, and the equation  $2xx + 8x + 16 = 400$  gives the lesser number of each pair; for if we make  $x = 12$ , and  $x + 4 = 16$ , the numbers  $12$  and  $16$  will solve the problem; on the other hand, if we make  $x = -16$ , we shall have  $x + 4 = -12$ , and the numbers  $-16$  and  $-12$  will equally solve the problem; for their difference is  $+4$ , and the sum of their squares  $+400$ : here then we may observe, that affirmative and negative solutions of problems are of equal estimation in the nature of things, though perhaps not amongst men, the narrowness of our minds contracting our views; but truth does justice alike to all: certainly negative numbers differ no more from affirmative ones, than affirmative ones do from one another, which is in degree, not in kind; and therefore, in the nature of things, negative quantities ought no more to be excluded

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 excluded out of the scale of number than affirmative  
 ones, though in common life they are set aside.

# P R O B L E M    72.

114. *What two numbers are those, whose sum is seven-  
 teen, and the sum of their cubes one thousand three  
 hundred forty-three?*

## S O L U T I O N.

For the two numbers sought put  $x$  and  $17 - x$ , and  
 the cube of the former will be  $xxx$ , and the cube of  
 the latter  $4913 - 867x + 51xx - xxx$ , as appears from  
 the following computation :

$$\begin{array}{r}
 17 - x \\
 17 - x \\
 \hline
 289 - 17x + xx \\
 \quad - 17x \\
 \hline
 289 - 34x + xx \\
 \quad 17 - x \\
 \hline
 4913 - 578x + 17xx - x^3 \\
 \quad - 289x + 34xx \\
 \hline
 4913 - 867x + 51xx - x^3.
 \end{array}$$

Therefore the sum of these two cubes will be  $51xx - 867x + 4913 = 1343$ , and the equations will be the  
 same, whichever of the two numbers sought  $x$  is  
 made to stand for ; but if  $51xx - 867x + 4913 = 1343$ ,  
 we shall have  $51xx - 867x = -3570$  ; divide the  
 whole by 51, which, though not necessary, is how-  
 ever convenient, to render the equation more simple,  
 since it may be done without fractions, and you will  
 have,  $xx - 17x = -70$  ; which, being reduced as in  
 art. 103, gives  $x = 7$ , or 10 ; therefore 7 and 10 are  
 the two numbers sought.

## P R O B L E M

## PROBLEM 73.

115. *Let there be a square whose side is a hundred and ten inches; it is required to assign the length and breadth of a rectangled parallelogram, or long square, whose perimeter shall be greater than that of the square by four inches, but whose area shall be less than the area of the square by four square inches.*

N. B. By the perimeter of a plain figure is meant the length of a line that will encompass it round; so that the perimeter of a square is equal to four times its side; and the perimeter of a rectangled parallelogram is equal to twice its length and twice its breadth added together.

## SOLUTION.

Since the side of the given square is 110 inches, its area will be 12100 square inches; therefore the area of the parallelogram sought will be 12096 square inches: again, the perimeter of the given square is 440 inches; therefore the perimeter of the parallelogram sought must be 444 inches; therefore half its perimeter, or its length and breadth added together, must be 222 inches; therefore, if either the length or breadth be called  $x$ , the other will be  $222 - x$ , and the area will be  $222x - xx = 12096$ ; which equation resolved according to art. 103, will give  $x = 96$ , or 126; therefore the breadth of the parallelogram sought must be 96 inches, and the length 126 inches: and these numbers will answer the conditions of the question; for twice the length will be 252, twice the breadth 192, and the whole perimeter 444; moreover  $126 \times 96$ , or the area, will be 12096, as the problem requires.

## SCHOLIUM.

This problem shews how grossly they are mistaken who think to estimate the areas or magnitudes of plain figures

figures by their perimeters, as if such figures were greater or less in proportion as their perimeters were 10; whereas here we see, that the perimeter of one figure may be greater than that of another by four inches, and at the same time its area may be less than the area of that other by four square inches. This error, it is true, does not obtain but in low and vulgar minds, nor there neither any longer than whilst it continues to be a matter of mere speculation, and truth and falsehood are equally indifferent to them: for whenever men come to apply their notions, and find it their interest not to be mistaken, then it is, and frequently not till then, that they begin to look about them, correct their errors, and entertain more just and accurate notions of things. The greatest part of mankind have a natural aversion to abstract thinking, and, where their interest is not concerned, will rather submit their opinions to humour, caprice, and custom, or be content to be without any opinions at all, than they will examine strictly into the nature of things.

#### PROBLEM 74.

116. *One buys a certain number of oxen for eighty guineas; where it must be observed, that if he had bought four more for the same money, they would have come to him a guinea apiece cheaper: What was the number of oxen?*

#### SOLUTION.

For the number of oxen put  $x$ ; then to find the price of a single ox, say, if  $x$  oxen cost 80 guineas, what will one ox cost? and the answer is  $\frac{80}{x}$ ; and for the same reason, if he had bought four more, that is,  $x+4$  for the same money, the price of an ox would have been  $\frac{80}{x+4}$ ; but, according to the problem, the latter

latter price is less than the former by one guinea; whence we have this equation  $\frac{80}{x} - 1 = \frac{80}{x+4}$ , therefore  $80 - x = \frac{80x}{x+4}$ ; therefore  $80 - x \times 4 + x$  or  $320 + 76x - xx = 80x$ ; therefore  $xx + 80x = 76x + 320$ ; therefore  $xx = -4x + 320$ . Here then  $A=1, B=-4, C=320, BB=16, 4AC=1280, ss=1296, s=36, \frac{B+s}{2A}=16, \frac{B-s}{2A}=-20$ ; therefore  $x=+16$ , or  $-20$ ; therefore the number of oxen was 16, the negative root  $-20$  having no place in this problem; and this number 16 answers the condition of the problem; for if 16 oxen cost 80 guineas, one will cost 5 guineas; but if 20 oxen cost 80 guineas, one will cost 4 guineas.

*N. B.* The equation  $\frac{80}{x} - 1 = \frac{80}{x+4}$ , gave  $x=+16$  or  $-20$ , not because the number  $-20$  would solve the problem, but because it would solve the equation; for if we make  $x=-20$ , we shall have  $\frac{80}{x} = -4$ , and  $\frac{80}{x} - 1 = -5$ ; on the other side, we shall have  $x+4 = -16$ , and  $\frac{80}{x+4} = -5$ ; therefore if  $x$  be made equal to  $-20$ , we shall have  $\frac{80}{x} - 1 = \frac{80}{x+4}$ , because both sides are equal to  $-5$ ; and so in all other cases we shall always find, that the several roots of an equation will be such as will equally solve that equation, though perhaps they may not be equally proper to solve the problem from whence the equation was deduced: but of this more in another place.

## PROBLEM 75.

117. *A certain company at a tavern had a reckoning of seven pounds four shillings to pay; upon which, two of the company breaking off, obliged the rest to pay one shilling apiece more than they should have done: What was the number of persons?*

## SOLUTION.

For the number of persons put  $x$ ; then, to find the number of shillings every man should have paid, say, if  $x$  persons were to have paid 144 shillings, what must one man have paid? and the answer is  $\frac{144}{x}$ ; therefore

$\frac{144}{x}$  is the number of shillings every man should have paid; and for the same reason  $\frac{144}{x-2}$  is the number of shillings every man did pay; but, according to the problem, this latter reckoning is greater than the former by one shilling; whence the equation will be

$$\frac{144}{x} + 1 = \frac{144}{x-2}; \text{ therefore } 144 + x = \frac{144x}{x-2}; \text{ therefore}$$

$$x-2 \times 144 + x, \text{ or } xx + 142x - 288 = 144x; \text{ therefore } xx - 288 = 2x; \text{ therefore } xx = 2x + 288. \text{ Here then}$$

$$A=1, B=2, C=288, BB=4, 4AC=1152, ss=1156,$$

$$s=34, \frac{B+s}{2A}=18, \frac{B-s}{2A}=-16; \text{ therefore } x=+18,$$

or  $-16$ ; but negative roots have no place in this sort of problems; therefore the number of persons was

$$18, \text{ which answers the condition; for } \frac{144}{18}=8, \text{ and}$$

$$\frac{144}{16}=9.$$

## PROBLEM 76.

118. *What number is that, which being added to its square root will make two hundred and ten?*

## SOLUTION.

For the number sought put  $xx$ ; then will its square root be  $x$ , and the equation will be  $xx + x = 210$ , or  $xx = -x + 110$ ; where  $A=1$ ,  $B=-1$ ,  $C=210$ ,  $BB=1$ ,  $4AC=840$ ,  $ss=841$ ,  $s=29$ ,  $\frac{B+s}{2A}=14$ ,  $\frac{B-s}{2A}=-15$ ; therefore  $x = +14$ , or  $-15$ ; therefore  $xx$  or the number sought equals 196 or 225, supposing the square root of 225 to be  $-15$ ; and either of these two numbers will answer the condition; for  $196 + 14 = 210$ , and  $225 - 15 = 210$ .

## PROBLEM 77.

119. *What two numbers are those, the product of whose multiplication is one hundred ninety-two, and the sum of whose squares is six hundred and forty?*

## SOLUTION.

For the two numbers sought put  $x$  and  $\frac{192}{x}$ ; then will the square of the former be  $xx$ , and that of the latter  $\frac{36864}{xx}$ , and the sum of their squares will be  $xx + \frac{36864}{xx} = 640$ ; which equation will be the same, whichever of the two numbers sought  $x$  is made to stand for; but if  $xx + \frac{36864}{xx} = 640$ , we shall have  $x^4 + 36864 = 640xx$ ; and  $x^4 = 640x^2 - 36864$ : here then  $A=1$ ,  $B=640$ ,  $C=-36864$ ,  $BB=409600$ ,  $4AC=-$   
I
147456,

$147456$ ,  $ss=262144$ ,  $s=512$ ,  $\frac{B+s}{2A}=576$ ,  $\frac{B-s}{2A}$   
 $=64$ ; therefore  $xx=576$ , or  $64$ ; therefore  $x=+$   
 or  $-24$ , or  $+$  or  $-8$ ; therefore the two numbers  
 sought are  $8$  and  $24$ .

### PROBLEM 78.

120. *One lays out a certain sum of money in goods,  
 which he sold again for twenty-four pounds, and  
 gained as much per cent. as the goods cost him: I de-  
 mand what they cost him?*

*N. B.* One's gain *per cent.* is so much as he gains,  
 every hundred pounds he lays out; or if he does not  
 lay out so much as a hundred pounds, his gain *per  
 cent.* however, is so much as he would have gained if  
 he had laid out a hundred pounds with the same ad-  
 vantage: thus if he lays out  $20$  pounds and gains  $2$   
 pounds, he is said to make  $10$  *per cent.* of his money,  
 because  $20$  pounds is to  $2$  pounds as  $100$  pounds is  
 to  $10$  pounds.

### SOLUTION.

Put  $x$  for the money laid out, and the gain will be  
 $24-x$ ; say then, by the golden rule, if in laying out  
 $x$  he gained  $24-x$ , what would he have gained if he  
 had laid out  $100$  pounds to the same advantage? and  
 the answer will be  $\frac{2400-100x}{x}$ ; therefore  $\frac{2400-100x}{x}$   
 will be his gain *per cent.*; but, according to the pro-  
 blem, this gain is equal to  $x$ , the money laid out;  
 therefore  $x=\frac{2400-100x}{x}$ , and  $xx=2400-100x$ :  
 here then  $A=1$ ,  $B=-100$ ,  $C=2400$ ,  $BB=10000$ ,  
 $4AC=9600$ ,  $ss=19600$ ,  $s=140$ ,  $\frac{B+s}{2A}=20$ ,  $\frac{B-s}{2A}$   
 $=-120$ ; therefore the money laid out was  $20$  pounds;  
 therefore his gain *per 20* was  $4$  pounds; therefore his  

O 2
gain

gain *per cent.* was 20 pounds, equal to the money laid out.

### PROBLEM 79.

121. One lays out thirty-three pounds fifteen shillings in cloth, which he sold again for forty-eight shillings per piece, and gained as much in the whole as a single piece cost : I demand how he bought in his cloth per piece ?

### SOLUTION.

Put  $x$  for the number of shillings every single piece was bought for, and the gain *per* piece will be  $48 - x$  ; say then, by the rule of proportion, if in laying out  $x$  he gained  $48 - x$ , what did he gain in laying out 33 pounds 15 shillings, or 675 shillings ? and the answer will be  $\frac{32400 - 675x}{x}$  ; therefore  $\frac{32400 - 675x}{x}$  will be his whole gain ; but, according to the problem, the whole gain was equal to  $x$ , the money given for a single piece ; therefore  $x = \frac{32400 - 675x}{x}$  ; therefore  $xx = 32400 - 675x$  ; therefore  $A = 1, B = -675, C = 32400, BB = 455625, 4AC = 129600, ss = 585225, s = 765, B \div s = 45, \frac{B - s}{2A} = -720$  ; therefore  $x = +45$ , or  $-720$  ; therefore the money every single piece was bought for was 45 shillings, and the gain *per* piece was 3 shillings ; but if 45 shillings gains 3 shillings, 33 pounds 15 shillings, or 675 shillings, will gain 45 shillings ; therefore the whole gain was 45 shillings, equal to the money given for a single piece.

N. B. It is not impossible but that sometimes two different problems may produce one and the same equation ; and then the equation must provide equally for both : therefore, in such a case, though the equation has two roots, and both affirmative, yet it must

not

not be expected that both roots should equally serve for the solution of one problem, and that there should be no solution left for the other; we ought rather to conclude, whenever an equation gives two roots, and both affirmative, whereof one only will solve the problem that produced the equation; we ought, I say, rather to conclude, that the other root is for the solution of some other problem producing the same equation; a curious instance whereof we have in the two following problems.

PROBLEM 80.

122. *Two travellers, A and B, set out from two places C and D at the same time, A from C bound for D, and B from D bound for C; when they met and had computed their travels, it was found, that A had travelled thirty miles more than B, and that, at their rate of travelling, A expected to reach D in four days, and B to reach C in nine days: I demand the distance between the two places C and D?*

SOLUTION.

Put  $x$  for the number of miles between C and D, then it is plain that A and B both together had travelled  $x$  miles when they met; therefore as much as the miles travelled by A exceeded  $\frac{x}{2}$ , just so much did the

miles travelled by B come short of  $\frac{x}{2}$ ; but, by the supposition, A's miles exceeded those of B by 30; therefore A must have travelled  $\frac{x}{2} + 15$  or  $\frac{x+30}{2}$  miles;

and B must have travelled  $\frac{x}{2} - 15$  or  $\frac{x-30}{2}$  miles;

therefore the remaining part of A's journey is  $\frac{x-30}{2}$  miles, which he expects to perform in four

days, and the remaining part of B's journey is  $\frac{x+30}{2}$  miles, which he expects to perform in 9

days : these things being allowed, let us now enquire into the number of days each hath travelled already ;

and first for *A* say, if *A* expects to travel  $\frac{x-30}{2}$  miles in 4 days, in how many days did he travel

$\frac{x+30}{2}$  miles? and the answer is  $\frac{4 \times \frac{x+30}{2}}{\frac{x-30}{2}} = \frac{4 \times x+30}{x-30}$ ;

then for *B* say, if *B* expects to travel  $\frac{x+30}{2}$  miles in 9 days, in how many days did he travel  $\frac{x-30}{2}$

miles? and the answer is  $\frac{9 \times x-30}{x+30}$  ; therefore *A*

hath travelled  $\frac{4 \times x+30}{x-30}$  days, and *B*  $\frac{9 \times x-30}{x+30}$

days from the first time of their setting out : but as they both set out at the same time, and are now met, they must both have travelled the same number of

days ; therefore  $\frac{4 \times x+30}{x-30} = \frac{9 \times x-30}{x+30}$  : multiply

both sides of the equation into  $x-30$ , and you will

have  $4 \times x+30 = \frac{9 \times x-30 \times x-30}{x+30}$  ; again multi-

ply by  $x+30$ , and you will have  $4 \times x+30 \times$

$x+30 = 9 \times x-30 \times x-30$  ; extract the square

root of both sides, and you will have  $\pm 2 \times x+30$

$= \pm 3 \times x-30$  : this general equation resolves itself into four particular ones, viz.

$$1^{\text{st}}, \quad +2 \times x+30 = +3 \times x-30.$$

$$2^{\text{d}}, \quad +2 \times x+30 = -3 \times x-30.$$

$$3^{\text{d}}, \quad -2 \times x+30 = +3 \times x-30.$$

$$4^{\text{th}}, \quad -2 \times x+30 = -3 \times x-30.$$

But

But as the two last of these equations give but the same values as the two former, I shall only make use of the two former, thus :

1st, Suppose  $+2 \times x + 30 = +3 \times x - 30$ , then we shall have  $2x + 60 = 3x - 90$ , and  $x = 150$ .

2dly, Suppose  $+2 \times x + 30 = -3 \times x - 30$ , then we shall have  $2x + 60 = -3x + 90$ , and  $x = 6$ ; therefore the distance between the two places *C* and *D* must either be 150 miles, or 6 miles; but 6 miles it cannot be, because when *A* came up to *B*, he had travelled 30 miles more than *B*, and had not yet reached *D*; therefore the distance between the two places *C* and *D* must be 150 miles; which will satisfy the problem; for then *A* must have travelled  $75 + 15$ , or 90 miles, and *B*  $75 - 15$ , or 60 miles, from the time of their setting out; therefore *A* has 60 miles, and *B* 90, to travel; but if *A* could travel 60 miles in 4 days, he must, at the same rate, have travelled 90 miles in 6 days; and if *B* could travel 90 miles in 9 days, he must have travelled 60 miles also in 6 days; therefore they both travelled the same number of days from the time of their first setting out to the time of their meeting, as the problem requires.

# PROBLEM 81.

123. *Two travellers A and B set out from two places C and D at the same time; A from C with a design to pass through D, and B from D with a design to travel the same way: after A had overtaken B, and they had computed their travels, it was found, that they had both together travelled thirty miles, that A had passed through D four days before, and that B, at his rate of travelling, was a nine days journey distant from C: I demand the distance between the two places C and D.*

## SOLUTION.

Put  $x$  for the number of miles from  $C$  to  $D$ ; then it is plain, that  $A$  must have travelled more miles than  $B$  by  $x$ ; but they both together travelled 30 miles, by the supposition; therefore as much as  $A$ 's miles exceeded 15, just so much  $B$ 's miles came short of 15: but the whole difference was  $x$ , as above; therefore  $A$  must have travelled  $15 + \frac{x}{2}$  or  $\frac{30+x}{2}$  miles, and  $B$  must have travelled  $15 - \frac{x}{2}$  or  $\frac{30-x}{2}$  miles; therefore  $A$ 's distance from  $D$ , after he had overtaken  $B$ , was  $\frac{30-x}{2}$  miles, which he had travelled in 4 days, and  $B$ 's distance from  $C$  was  $\frac{30+x}{2}$  miles, which by the problem he could travel in 9 days; therefore, to find how many days each had travelled already, say, if  $A$  hath travelled  $\frac{30-x}{2}$  miles from  $D$  in 4 days, in how many days did he travel  $\frac{30+x}{2}$  miles since his departure from

$C$ ? and the answer is  $4 \times \frac{\frac{30+x}{2}}{\frac{30-x}{2}} = 4 \times \frac{30+x}{30-x}$ ; again

say, if  $B$  could travel  $\frac{30+x}{2}$  miles, the whole distance from  $C$ , in 9 days, in how many days did he travel  $\frac{30-x}{2}$  miles since his setting out from  $D$ ? and the

answer is  $\frac{9 \times \frac{30-x}{2}}{\frac{30+x}{2}}$ ; but as they both set out at the same

same time, and *A* has now overtaken *B*, they must both have travelled the same number of days; therefore we have this equation,  $\frac{4 \times 30 + x}{30 - x} = \frac{9 \times 30 - x}{30 + x}$ :

multiply both sides into  $30 - x$ , and you will have  $4 \times 30 + x = \frac{9 \times 30 - x \times 30 - x}{30 + x}$ ; again multiply

by  $30 + x$ , and you will have  $4 \times 30 + x \times 30 + x = 9 \times 30 - x \times 30 - x$ ; but the product of  $30 - x \times 30 - x$  differs nothing from the product of  $x - 30 \times x - 30$ , as will appear upon tryal, and will be fur-

ther evident from hence, that  $30 - x$  and  $x - 30$  differ no more from one another than an affirmative quantity does from an equal negative one, and therefore each multiplied into itself must give the same product, therefore the equation, as it now stands, is,  $4 \times x + 30 \times x + 30 = 9 \times x - 30 \times x - 30$ ; but this equation is the same with the equation deduced from the last problem, which justifies what I observed before, art. 121, that different problems may produce the same equation; therefore the two roots of this equation will be 6 and 150, as in the last article; therefore the distance between the two places *C* and *D* must either be 6 miles, or 150 miles; but 150 miles it cannot be, because, after *A* had passed from *C* beyond *D*, and at last had overtaken *B*, they had both travelled but 30 miles; therefore the distance from *C* to *D* must be 6 miles; and this number will answer the conditions of the problem; for then *A*, when he had overtaken *B*, had travelled  $15 + 3$  or 18 miles, and *B*  $15 - 3$  or 12 miles; therefore *A* had got 12 miles beyond *D* in 4 days time, and *B* was 18 miles distant from *C*, which he could travel in 9 days; but at the rate of 12 miles in 4 days, *A* must have performed his 18 miles journey in 6 days; and at the rate of 18 miles in 9 days, *B* must have performed his 12 miles journey also in 6 days; therefore, from the time of their first setting

out to the time of *A*'s overtaking *B*, they had both travelled the same number of days, as the problem requires; therefore the supposition whereupon this calculation was founded, to wit, that the distance of *C* from *D* was 6 miles, is just.

*N. B.* The solutions here given of the two last problems are, in my opinion, the most natural, though somewhat different from the rest.

### A L E M M A.

124. *The sum of a series of quantities in arithmetical progression may be had by adding the greatest and least terms together, and then multiplying either half that sum by the whole number of terms, or the whole sum by half the number of terms; or, lastly, by multiplying the whole sum into the whole number of terms, and then taking half the product: thus in the series 2, 4, 6, 8, 10, 12, where the least term is 2, the greatest 12, their sum 14, and the number of terms 6; the sum of all the terms taken together will be  $7 \times 6$ , or  $14$*

*$\times 3$ , or  $\frac{14 \times 6}{2} = 42$ . This will best appear by writing down the series 2, 4, 6, 8, 10, 12, and then by writing down over it the same series inverted, 12, 10, 8, 6, 4, 2: for, if this be done, 2, the first term of the lower series, added to 12, the first term of the upper series (which is the same as the greatest and least terms of the same series added together) will make 14; in like manner, every term of the lower series added to the next above it will make 14; therefore both the serieses together will be equal to 14 as often taken as there are terms in either series, that is, 6 times 14, or 84; therefore either series taken alone will be equal to 42.*

12	10	8	6	4	2
2	4	6	8	10	12
14	14	14	14	14	14

The design of this lemma is, to add the terms of a series together, where only the greatest and least terms and the number of terms are known, or supposed to be known; the intermediate terms being either not assigned, or too many to be summed up by a continual addition.

### PROBLEM 82.

125. *A traveller, as A, sets out from a certain place, and travels one mile the first day, two miles the second day, three the third, four the fourth, &c.; and five days after, another, as B, sets out from the same place, and travels the same road at the rate of twelve miles every day: I demand how long and how far A must travel before he is overtaken by B.*

### SOLUTION.

Put  $x$  for the number of days  $A$  travelled before he was overtaken by  $B$ ; then, to find an expression for the number of miles travelled by him in that time, I observe that in three days  $A$  travelled over  $1+2+3$  miles, that is, he travels over a series of miles in arithmetical progression, whereof the number of terms is 3, the greatest term 3, and the least term 1; in four days he travels over a series whereof the number of terms is 4, the greatest term 4, and the least 1; therefore, universally, in any number  $x$  of days, he must travel over a series of miles in arithmetical progression, whereof the number of terms is  $x$ , the greatest term  $x$ , and the least term 1; but the sum of the extremes of this series is  $x+1$ , which, multiplied by  $x$ , the number of terms, gives  $xx+x$ , the half whereof is  $\frac{xx+x}{2}$ ; therefore, by the lemma foregoing,  $\frac{xx+x}{2}$  will be the sum of this series, and consequently the miles travelled by  $A$  before he was overtaken: again,

again, if  $A$  travel  $x$  days,  $B$  must have travelled  $x - 5$  days, which at the rate of 12 miles a day, gives  $12x - 60$  for the miles travelled by  $B$  when he overtook  $A$ ; but as they both set out from the same place, and are now got together, they must have travelled the same number of miles; whence we have

this equation,  $\frac{xx + x}{2} = 12x - 60$ ; therefore  $xx + x = 24x - 120$ ; therefore  $xx = 23x - 120$ ; compare this equation with the general one in art. 103, and you will have  $A = 1$ ,  $B = 23$ ,  $C = -120$ ,  $BB = 529$ ,  $4AC = -480$ ,  $ss = 49$ ,  $s = 7$ ,  $\frac{B+s}{2A} = 15$ ,  $\frac{B-s}{2A} = 8$ ; there-

fore  $x = 8$ , or  $15$ ; now, for the better application of these roots to the solution of this problem, it must be observed, that the problem is more limited than the equation deduced from it; just as if, in translating out of one language into another, the terms of the latter, instead of being adequate to those of the former, should be found to be of a more extensive signification: in the problem it is only supposed that  $B$  overtakes  $A$ , whereas in the equation it is supposed that  $A$  and  $B$  are got both together by having travelled the same number of miles from their first setting out, without specifying whether this arises from  $B$ 's overtaking  $A$ , or from  $A$ 's overtaking  $B$ ; both which in this case must necessarily happen in the course of their travels, provided they be but continued long enough for that purpose: for since at first  $B$  is the swifter traveller, whenever they come together, it must arise from  $B$ 's overtaking  $A$ , which happens after  $A$  has travelled 8 days; then, if we suppose them still to continue their travels,  $B$  passes by  $A$ , and continues before him for some time; but after 12 days,  $A$  becomes the swifter traveller, and must necessarily come up to  $B$  again after he has travelled 15 days: therefore though the two roots, 8 and 15, will both answer the condition of the equation, yet but one of them,

them, to wit, 8, will answer the condition of the problem; and that both of them will answer the condition of the equation, will be evident as follows.

In 8 days *A* travels over a series of miles whereof the number of terms is 8, the greatest 8, and the least 1; the sum of which series is 36 miles; but when *A* has travelled 8 days, *B* must have travelled 3 days, during which time, at the rate of 12 miles a day, he also must have travelled 36 miles; therefore after *A* hath travelled 8 days, *A* and *B* must necessarily find themselves together: again, in 15 days, *A* must have travelled over a series of miles, whereof the number of terms is 15, the greatest 15, the least 1, and the sum 120 miles; but when *A* had travelled 15 days, *B* must have travelled 10 days, which at 12 miles a day gives also 120 miles; therefore now again *A* and *B* must find themselves together; and consequently 8 and 17 equally answer the supposition contained in the equation.

*N. B.* If we suppose *B* after 5 days to have begun to follow *A*, and to have travelled only 10 miles a day, he could never have overtaken *A*, nor *A* him, so that in this case both the roots would have become impossible, as will be found by the resolution of an equation founded upon this supposition.

### P R O B L E M 83.

126. *It is required to divide the number ten into two such parts, that the product of their multiplication being added to the sum of their squares, may make seventy-six.*

### S O L U T I O N.

The two parts sought,  $x$  and  $10 - x$ .

The product of their multiplication,  $10x - xx$ .

The sum of their squares,  $2xx - 20x + 100$ .

The product of their multiplication added to the sum of their squares,  $\left. \begin{array}{l} \\ \end{array} \right\} x^2 - 10x + 100 = 76$ .

Whence

Whence  $x=4$ , or 6; but this equation will be the same, which part soever  $x$  is put for; therefore the two parts sought are 4 and 6.

### PROBLEM 84.

127. *It is required to find two numbers with the following properties, to wit, that twice the first with three times the second may make sixty, and moreover, that twice the square of the first with three times the square of the second may make eight hundred and forty.*

### SOLUTION.

For the two numbers sought put  $x$  and  $y$ , and we shall have

$$\text{Equ. 1st, } 2x + 3y = 60, \text{ and}$$

$$\text{Equ. 2d, } 2x^2 + 3y^2 = 840.$$

From the first equation,  $2x + 3y = 60$ , we have

$$\text{Equ. 3d, } x = \frac{60 - 3y}{2}; \text{ and by squaring}$$

both sides we have

$$\text{Equ. 4th, } xx = \frac{3600 - 360y + 9yy}{4}.$$

From the second equation,  $2xx + 3yy = 840$ , we have

$$\text{Equ. 5th, } xx = \frac{840 - 3yy}{2}.$$

Compare the two values of  $xx$  in the fourth and fifth equations, which must necessarily be equal one to the other, and you will have  $\frac{3600 - 360y + 9yy}{4} =$

$\frac{840 - 3yy}{2}$ ; multiply both sides into 2, by halving the

denominators, and you will have  $\frac{3600 - 360y + 9yy}{2}$

$= 840 - 3yy$ ; therefore  $3600 - 360y + 9yy = 1680 - 6yy$ ; therefore  $3600 - 360y + 15yy = 1680$ ; therefore  $15yy - 360y = -1920$ ; therefore  $15yy = 360y - 1920$ ;

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1920; divide by 15 for a more simple equation, and you will have  $yy = 24y - 128$ ; whence  $y = 8$ , or 16: suppose  $y = 8$ , then since by the third equation  $x = \frac{60 - 3y}{2}$ , we shall have  $x = 18$ ; suppose  $y = 16$ , then

we shall have  $x$  or  $\frac{60 - 3y}{2} = 6$ ; therefore there are

two pair of numbers that will equally answer the conditions of this problem, to wit, 18 and 8, and also 6 and 16: for a proof, let us first suppose the numbers to be 18 and 8; and we shall have twice the first number with three times the second  $= 36 + 24 = 60$ ; and twice the square of the first together with three times the square of the second equal to  $648 + 192 = 840$ : secondly, let us suppose the numbers to be 6 and 16; and we shall have twice the first with three times the second equal to  $12 + 48 = 60$ ; and twice the square of the first with three times the square of the second equal to  $72 + 768 = 840$ .

### PROBLEM 85.

128. *To find four numbers in continual proportion, and such, that the sum of the two middle terms may be eighteen, and that of the extremes twenty-seven.*

*Note,* Four numbers are said to be in continual proportion, when the first is to the second as the second is to the third, and the second is to the third as the third is to the fourth.

### SOLUTION.

For the two middle terms put  $x$  and  $y$ , without intending which is to be the greater; then the extreme next to  $x$  may be found by saying, as  $y$  is to  $x$

so is  $x$  to  $\frac{xx}{y}$ , and the extreme next to  $y$  may be

found by saying, as  $x$  is to  $y$ , so is  $y$  to  $\frac{yy}{x}$ ; therefore

the

the extremes are  $\frac{xx}{y}$  and  $\frac{yy}{x}$ , and their sum  $\frac{x^3+y^3}{xy}$ ; therefore the fundamental equations are 1st,  $x+y=18$ , or  $x=18-y$ ; and 2dly,  $\frac{x^3+y^3}{xy}=27$ , or  $x^3+y^3=27xy$ ; instead of  $x$  in this equation put  $18-y$ , its value in the last, and you will have  $x^3=5832-972y+54y^2-y^3$ ; therefore  $x^3+y^3=5832-972y+54yy$ ; you will also have  $27xy$  or  $27y \times 18-y=486y-27yy$ ; therefore  $5832-972y+54yy=486y-27yy$ ; transpose  $486y-27yy$ , and you will have  $81yy-1458y+5832=0$ ; divide all by 81, which may be done without a fraction, and you will have  $yy-18y+72=0$ ; which equation being resolved, either by the general theorem, or any other way, gives  $y=6$ , or  $12$ ; and since the equation will be the same, whichever of the two middle terms  $y$  stands for, it follows, that the two middle terms are 6 and 12; whence the extreme next to 6 is 3, and that next to 12 is 24; and the numbers are either 3, 6, 12, and 24, or 24, 12, 6, and 3, for either way they will answer the conditions of the problem.

### PROBLEM 86.

129. *There are three numbers in continual proportion, whose sum is nineteen, and the sum of their squares one hundred thirty-three: What are the numbers?*

### SOLUTION.

For the three numbers sought put  $x, y$ , and  $z$ ; then since, by the first condition,  $x$  is to  $y$  as  $y$  is to  $z$ , by multiplying extremes and means we have  $yy=xz$ ; again, by the second condition of the problem, we have  $x+y+z=19$ , and  $19-y=x+z$ , and (squaring both sides)  $361-38y+yy=xx+2xz+zz$ ; subtract  $yy$  from one side of the equation, and its equal  $xz$  from the other, and you will have  $361-38y=x^2+xz$

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$xz + z^2 = x^2 + y^2 + z^2 = 133$  by the third condition of the problem : having thus expunged both  $x$  and  $z$  at once, resolve the equation  $361 - 38y = 133$ , and you will have  $y$  the middle term equal to 6, and  $19 - y$ , or the sum of the extremes,  $= 13$ ; therefore the problem produced is now reduced to this, *viz.* Of three numbers in continual proportion, whereof six, the middle term, and thirteen, the sum of the extremes, are given to find the extremes : this problem is of the same nature with that in art. 112, and, being resolved, gives 4 and 9 for the extremes; therefore the three numbers sought, are 4, 6, and 9, or 9, 6, and 4.

### PROBLEM 87.

130. *To find two numbers such, that their difference multiplied into the difference of their squares shall make thirty-two, but their sum multiplied into the sum of their squares shall make two hundred seventy-two.*

### SOLUTION.

For the two numbers sought put  $x$  and  $y$ ; and the first fundamental equation will be  $\frac{x-y}{x+y} \times \frac{x^2-y^2}{x^2+y^2}$ , or  $\frac{x-y}{x+y} \times \frac{x-y}{x+y} \times x+y$ , or  $x^2 - 2xy + y^2 \times x+y = 32$ ; therefore

$$\text{Equ. 1st, } x^2 - 2xy + y^2 = \frac{32}{x+y}.$$

The second fundamental equation is,  $\frac{x+y}{x+y} \times \frac{x^2+y^2}{x^2+y^2} = 272$ ; therefore

$$\text{Equ. 2d, } x^2 + y^2 = \frac{272}{x+y}.$$

From twice the second equation  
subtract the first, that is, from

$$2x^2 + 2y^2 = \frac{544}{x+y}$$

$$\text{subtract } x^2 - 2xy + y^2 = \frac{32}{x+y}$$

$$\text{and you will have } x^2 + 2xy + y^2 = \frac{512}{x+y},$$

that

that is,  $\overline{x+y}^2 = \frac{512}{x+y}$ ; therefore  $\overline{x+y}^3 = 512$ , and  $x+y$

$= \sqrt[3]{512}$ , or the cube root of  $512 = 8$ : thus we have got the sum of the two numbers fought, to wit, 8; whence their difference may be found by the first

equation, thus;  $x^2 - 2xy + yy = \frac{3^2}{x+y}$ , that is,  $\overline{x-y}^2$

$= \frac{3^2}{8} = 4$ ; therefore  $x-y$ , or the difference of the

two numbers fought, equals 2; therefore the problem proposed is now reduced to this: *Having given eight the sum, and two the difference of the two numbers x and y, to find those numbers*; and by art. 26 we shall have  $x=5$ , and  $y=3$ ; which numbers will answer the conditions of the question.

*N. B.* After we had found  $x+y$ , the sum of the numbers equal to 8, we might have found the sum of their squares by the second equation, which gave  $x^2 +$

$y^2 = \frac{272}{x+y} = \frac{272}{8} = 34$ ; and then the problem would

have been reduced to this: *What two numbers are those, whose sum is eight, and the sum of their squares thirty-four?* which would have produced a quadratic equation, as in art. 113, whose two roots would have been 5 and 3, as before.

#### PROBLEM 88.

131. *To find two numbers such, that their difference added to the difference of their squares may make fourteen, and their sum added to the sum of their squares may make twenty-six.*

#### SOLUTION.

For the two numbers fought put  $x$  and  $y$ , and you will have the two following equations;

$$\text{Equ. 1st, } x - y + x^2 - y^2 = 14.$$

$$\text{Equ. 2d, } x + y + x^2 + y^2 = 26.$$

Add

Add these two equations together, and you will have  $2xx + 2x = 40$ ,  $xx + x = 20$ , and  $x = +4$ , or  $-5$ ; again, subtract the first equation from the second, and you will have  $2yy + 2y = 12$ ,  $yy + y = 6$ , and  $y = +2$ , or  $-3$ ; and as these two values of  $y$  were obtained without any manner of dependence upon those of  $x$ , it is plain that either of the values of  $x$  may be joined with either of the values of  $y$ ; and so we have no fewer than four pairs of numbers which will equally satisfy the conditions of the equations, to wit,  $+4$  and  $+2$ ,  $+4$  and  $-3$ ,  $-5$  and  $+2$ ,  $-5$  and  $-3$ ; but it is the first pair only, which, consisting of affirmative numbers, is proper for the solution of the problem, thus: the difference of 4 and 2 is 2, the difference of their squares 12, and  $2 + 12 = 14$ ; again, the sum of 4 and 2 is 6, the sum of their squares 20, and  $6 + 20 = 26$ . Let us see however how the other pairs will satisfy the conditions of the equations; make then  $x$  equal to 4,  $y$ , that is,  $+y = -3$ , and you will have  $-y = +3$ ; whence  $x - y = 4 + 3 = 7$ ,  $x^2 - y^2 = 16 - 9 = 7$ , and  $7 + 7 = 14$ ; again,  $x + y = 4 - 3 = 1$ , and  $x^2 + y^2 = 16 + 9 = 25$ , and  $1 + 25 = 26$ : in the next place, make  $x = -5$ , and  $y = +2$ , then we shall have  $x - y = -5 - 2 = -7$ ,  $x^2 - y^2 = 25 - 4 = 21$ , and  $-7 + 21 = 14$ ; again,  $x + y = -5 + 2 = -3$ , and  $x^2 + y^2 = 25 + 4 = 29$ , and  $-3 + 29 = 26$ : lastly, make  $x = -5$ , and  $y = -3$ , and you will have  $x - y = -5 + 3 = -2$ , and  $x^2 - y^2 = 25 - 9 = 16$ , and  $-2 + 16 = 14$ ; again,  $x + y = -5 - 3 = -8$ , and  $x^2 + y^2 = 25 + 9 = 34$ , and  $-8 + 34 = 26$ .

PROBLEM 89.

132. *What two numbers are those, whose sum, when added together, is equal to their product when multiplied together; and this sum or product, when added to the sum of their squares, makes twelve?*

## SOLUTION.

For the two numbers sought put  $x$  and  $y$ , and the fundamental equations will be 1<sup>st</sup>,  $x+y=xy$ ; and secondly,  $x+y+x^2+y^2=12$ : in the first of these fundamental equations, where  $x+y=xy$ , we have  $yx-x=y$ ; but  $yx-x$  is the product of  $y-1 \times x$ , or of  $x \times y-1$ ; therefore  $x \times y-1=y$ , and  $x=\frac{y}{y-1}$ ; but if instead of  $x$ , this value be substituted into the second fundamental equation, the equation will rise to a bi-quadratic, for the resolution whereof no rules have hitherto been given; therefore, to extricate ourselves out of this difficulty, it will be proper to have recourse to some other artifice, by trying other positions, as thus; for the sum of the two numbers sought put  $z$ ; then will  $z$  be also the product of their multiplication, by the supposition; and since this product  $z$  added to the sum of their squares gives 12, the sum of their squares will be  $12-z$ ; but every one knows, that if to the sum of the squares of any two numbers be added their double product, there will arise the square of their sum; therefore  $12-z+2z$ , or  $12+z=z^2$ ; which equation being resolved, gives  $z=+4$ , &c.; and therefore the question is now reduced to this: *What two numbers are those, whose sum is four, and the product of whose multiplication is four?* for the numbers sought, put  $x$  and  $4-x$ , and you will have  $4x-xx=4$ ; and changing the signs,  $xx-4x=-4$ ; and compleating the square,  $xx-4x+4=0$ ; and extracting the square root,  $x-2=\pm 0$ ; whence  $x=2$ , or 2, for the roots of this equation are equal; therefore 2 and 2 are the numbers desired in the question; and they will answer the conditions; for in the first place,  $2+2=4=2 \times 2$ ; and in the next place, 4 the sum of 2 and 2, being added to 8, the sum of their squares, gives 12.

COROL-

C O R O L L A R Y.

From our first attempt to solve this problem we may learn thus much however, that if any number whatever be made equal to  $y$ , then these two numbers  $y$  and  $\frac{y}{y-1}$  will always have this property, that their sum when added together will be equal to their product when multiplied together; thus if  $3=y$ , and consequently  $\frac{3}{2}=\frac{y}{y-1}$ , we shall have  $3+\frac{3}{2}=4\frac{1}{2}$ , and  $3\times\frac{3}{2}$  or  $\frac{9}{2}=4\frac{1}{2}$ ; whence it follows, that this problem cannot be solved in whole numbers in any other case than that we have here put.

\* P R O B L E M 90.

133. *What two numbers are those, whose sum added to the product of their multiplication makes thirty-four, and the same sum subtracted from the sum of their squares leaves forty-two?*

S O L U T I O N.

Here, to avoid all difficulties that would otherwise arise, put  $z$  for the sum of the two numbers sought; then, since this sum added to the product of their multiplication makes 34, the product of their multiplication will be  $34-z$ ; but this sum  $z$  subtracted from the sum of their squares, leaves 42; therefore the sum of their squares is  $42+z$ ; to this add their double product  $68-2z$ , and you will have  $110-z=z^2$ ; whence  $z=+10$ , &c. and  $34-z=24$ ; therefore now the question is, *What two numbers are those, whose sum is ten, and the product of their multiplication twenty-four?* and by art. 111, the two numbers sought are 4 and 6.

Whoever would see more questions of this nature, may consult *Bachet's* comment upon the 33d question of the first book of *Diophantus's* Arithmetics.

N. B. Having now done with quadratic equations, at least for a time, it may perhaps be expected that, according to order of method I should proceed on to equations of higher forms: but I shall take the liberty for once to dispense with that method; not but that I intend (God willing) to treat fully and distinctly of these equations hereafter; but in the mean time I think it more adviseable to employ the reader's thoughts in some other things, which I take to be of much greater importance, and more proper for his information.

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T H E

E L E M E N T S   O F   A L G E B R A .

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B O O K   I V .

*Of general Problems, and general Theorems deduced from them ; together with the Manner of applying and demonstrating these Theorems synthetically.*

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*The Design of this fourth Book more fully explained.*

Art. 134. **H**ITHERTO my young Analyst has been indulged for the most part in a sort of mixt Algebra, where letters were put only for unknown quantities: but if he would reason abstractedly upon his problems, and draw general conclusions from them, he must put letters not only for his unknown quantities, but also for such as are known; and so propose and solve his problems indefinitely. By this means, in the first place, he will obtain indefinite answers, which in many cases are much preferable to more particular ones, as they suit and solve all particular cases to which they are applicable; and in the next place he will be able to prove his work synthetically; which will not only confirm his former *analysis*,

but will also further inure and reconcile him to the operations of symbolical or specious Arithmetic; and so render him entire master of this sort of computation. A sufficient specimen of this sort of reasoning, both in the analytical and synthetical way, has already been given in our general theorem for the resolution of a quadratic equation, so that no more needs be said by way of preparation; it remains therefore now, that we look back upon some of the problems already solved, and shew how to solve them over again in general terms, as follows:

PROBLEM I. (See art. 26.)

135. *What two numbers are those, whose sum is  $a$ , and difference  $b$ ?*

SOLUTION.

Put  $x$  for the less number; then will the greater be  $x+b$ , and their sum  $2x+b=a$ ; whence  $2x=a-b$ , and  $x$  (the less number) will be  $\frac{a-b}{2}$ ; whence  $x+b$ , (the greater number) will be  $\frac{a-b}{2} + \frac{b}{1} = \frac{a-b+b}{2} = \frac{a+b}{2}$ ; so the greater number is found to be  $\frac{a+b}{2}$ ,

and the less  $\frac{a-b}{2}$ ; where  $a$  and  $b$  are left undetermined till some particular case of this problem is proposed to be compared with the general one; and then the quantities  $a$  and  $b$  will not only be determined in that case, but the problem may be solved by the general theorem without any further *analysis*. As for example, let it be proposed, as in art. 26, to find two numbers whose sum is 48, and difference 14: here it is plain that  $a$  in the general problem answers to 48 in the particular case, and  $b$  to 14; whence  $\frac{a+b}{2}$  (or the greater number)  $= \frac{48+14}{2} = \frac{62}{2} = 31$ ,  
and

and  $\frac{a-b}{2}$  (or the less number)  $= \frac{48-14}{2} = \frac{34}{2} = 17$ , so that the numbers sought are 31 and 17; which will answer the conditions of the question. Again, suppose we were to find two numbers whose sum is 35, and whose difference is 9: in this case it is plain that  $a$  and  $b$  have other significations; for here  $a=35$ , and  $b=9$ , and therefore  $\frac{a+b}{2}$  (or the greater number) will be 22, and  $\frac{a-b}{2}$  (or the less number) will be 13.

These theorems are capable of being translated out of Algebraic language into any other; though to no great purpose that I know of, to such as understand any thing of symbolical Arithmetic; for, in my opinion, they appear much more distinct as they are, and less liable to ambiguity. The foregoing problem, together with the answer belonging to it, being translated into common English, will stand thus:

# P R O B L E M.

*It is required, having given the sum and difference of any two numbers, to find the numbers themselves.*

*Ans. 1st. Add the difference to the sum, and half the aggregate will be the greater number. 2dly, Subtract the difference from the sum, and half the remainder will be the less number.*

That this is a true translation, is plain: for what is  $\frac{a+b}{2}$  but half the aggregate of the sum and difference added together? and what is  $\frac{a-b}{2}$  but half the remainder, after the difference is subtracted from the sum?

We come now, in the last place, to examine this theorem as it stands in general terms, and to try whether

ther it will answer the conditions of the problem in the letters themselves. It was proposed to find two numbers, whose sum is  $a$ , and whose difference is  $b$ ; and the answer was, that the greater number was  $\frac{a+b}{2}$ , and the less  $\frac{a-b}{2}$ : now that this is a true answer, will be evident from a bare addition and subtraction of the numbers themselves, without any other principles; for if  $\frac{a+b}{2}$  be added to  $\frac{a-b}{2}$ , their sum will be  $\frac{2a}{2}$  or  $a$ , which answers the first condition of the problem; and if  $\frac{a-b}{2}$  be subtracted from  $\frac{a+b}{2}$ , the remainder will be  $\frac{2b}{2}$  or  $b$ , which answers the second condition.

This is that which is called a *synthetical demonstration*, and doubtless shews the truth of the theorem to which it belongs, as well as the *analysis* whereby that theorem was investigated; but not so much to the satisfaction of the mind: for a *synthetical demonstration* only shews that a proposition is true; whereas an *analytical* one shews not only that a proposition is true, but why it is so; places you in the condition of the inventor himself, and unveils the whole mystery. *Synthetical demonstrations* usually require fewer principles than *analytical* ones, as will evidently appear, by comparing both, in this very example; and this I take to be the reason why the ancients, generally speaking, chose to demonstrate their propositions this way; not with a design to conceal their *analysis*, as some have, unjustly enough, imagined; but because this sort of demonstration required fewer principles to proceed upon, and those too such as were commonly known.

## PROBLEM 2.

136. *What three numbers are those, whereof the sum of the first and second is a, that of the first and third b, and that of the second and third c?*

## SOLUTION.

Put  $x$  for the first number sought; then will the second number be  $a - x$ , because the first and second numbers together make  $a$ ; for a like reason the third number will be  $b - x$ , because the first and third together make  $b$ : add now the second and third numbers together, and you will have  $a + b - 2x = c$ ; therefore  $2x + c = a + b$ ; therefore  $2x = a + b - c$ ; and  $x$  (or the first number)  $= \frac{a + b - c}{2}$ ; subtract now the first

number  $\frac{a + b - c}{2}$  from  $a$ , or which is all one, add  $\frac{-a - b + c}{2}$  to  $a$ , and you will have the second num-

ber equal to  $\frac{-a - b + c}{2} + \frac{a}{1} = \frac{-a - b + c + 2a}{2} = \frac{a - b + c}{2}$ ; again, subtract the first number  $\frac{a + b - c}{2}$

from  $b$ , and you will have the third number equal to  $\frac{-a - b + c}{2} + \frac{b}{1} = \frac{-a + b + c}{2}$ ; and thus we have all the three numbers sought, to wit,

$$\text{The first, } \frac{a + b - c}{2},$$

$$\text{The second, } \frac{a - b + c}{2},$$

$$\text{The third, } \frac{-a + b + c}{2},$$

To apply this general solution to some particular case, I shall make use of that in art. 42, where it was required

required to find three such numbers, that the sum of the first and second may make 60, that of the first and third 80, and that of the second and third 92: in this case it is plain that  $a=60$ ,  $b=80$ , and  $c=92$ ;

therefore  $\frac{a+b-c}{2}$  or the first number will be 24;

$\frac{a-b+c}{2}$  or the second number will be 36; and

$\frac{-a+b+c}{2}$  or the third number will be 56; which

numbers upon trial will be found to be such as the problem requires. But that the theorems here given are not only true in this particular case, but are universally so, will best appear from the synthetical demonstration following.

1st, The first number  $\frac{a+b-c}{2}$ , and the second number  $\frac{a-b+c}{2}$  being added together make  $\frac{2a}{2}$  or  $a$ , according to the first condition, the other quantities destroying one another.

2dly, The first number  $\frac{a+b-c}{2}$ , and the third number  $\frac{-a+b+c}{2}$  being added together make  $\frac{2b}{2}$  or  $b$ , according to the second condition.

Lastly, The second number  $\frac{a-b+c}{2}$  and the third number  $\frac{-a+b+c}{2}$  being added together make  $\frac{2c}{2}$  or  $c$ , according to the third condition.

This problem may also be solved somewhat more elegantly thus: put  $s$  for the unknown sum of all the three numbers sought: then if  $c$ , the sum of the second and third numbers, be subtracted from  $s$ , the sum of all three, there will remain the first number equal

equal to  $s - c$ ; in like manner  $b$ , the sum of the first and third numbers, subtracted from  $s$ , the sum of all three, leaves the second number equal to  $s - b$ ; and  $a$ , the sum of the first and second numbers, subtracted from  $s$ , the sum of all three, leaves the third number equal to  $s - a$ ; add now all these three numbers together, to wit,  $s - c$ ,  $s - b$ , and  $s - a$ , and the sum will be  $3s - a - b - c$ ; but the sum is  $s$ , by the supposition; therefore,  $3s - a - b - c = s$ ; and  $s = \frac{a + b + c}{2}$ , whence we have the following theorem:

Make  $\frac{a + b + c}{2} = s$ ; then if the numbers  $a$ ,  $b$ , and  $c$ , be taken backwards, and subtracted severally from  $s$ , the three remainders  $s - c$ ,  $s - b$ , and  $s - a$ , will be the three numbers sought, in order as they are supposed in the problem. Thus if  $a = 60$ ,  $b = 80$ , and  $c = 92$ , as before, we shall have  $\frac{a + b + c}{2}$  or  $s = 116$ ; whence the first number will be  $116 - 92$  or  $24$ , the second  $116 - 80$  or  $36$ , and the third  $116 - 60$  or  $56$ .

#### SCHOLIUM.

*What three numbers are those, whereof the product of the first and second is  $a$ , that of the first and third  $b$ , and that of the second and third  $c$ ?*

#### SOLUTION.

Put  $p$  for the product of all the three numbers; then since  $c$  is the product of the two last, we shall have the first number equal to  $\frac{p}{c}$ ; for a like reason the second equals  $\frac{p}{b}$ , and the third equals  $\frac{p}{a}$ ; and the product of all three equals  $\frac{p^3}{abc} = p$ ; therefore  $p^2 = abc$ , and  $p = \sqrt{abc}$ .

DEMON-

## DEMONSTRATION.

$\frac{p}{c} \times \frac{p}{b}$ , or the product of the first and second numbers, is,  $\frac{p^2}{bc} = \frac{abc}{bc} = a$ : and so of the rest.

## PROBLEM 3.

137. *It is required to find two numbers whose difference is  $b$ , and the difference of whose squares is  $a$ .*

## SOLUTION.

Put  $x$  for the less number, and consequently  $x+b$  for the greater; then will the square of the less number be  $xx$ , that of the greater  $xx+2bx+bb$ , and the difference of their squares  $2bx+bb=a$ ; therefore

$2bx=a-bb$ , and  $x$  (the less number)  $= \frac{a-bb}{2b}$ ; whence

$$x+b \text{ (the greater)} = \frac{a-bb}{2b} + \frac{b}{1} = \frac{a-bb+2bb}{2b} = \frac{a+bb}{2b}.$$

To apply this general solution, let it be required to find two numbers whose difference is 4, and the difference of whose squares is 112: here  $a=112$ ,  $b=4$ ,

$bb=16$ ,  $\frac{a-bb}{2b} = 12$ ,  $\frac{a+bb}{2b} = 16$ ; therefore the numbers are 12 and 16. The general demonstration

is as follows: if the less number  $\frac{a-bb}{2b}$  be subtracted

from the greater  $\frac{a+bb}{2b}$ , their difference will be  $\frac{2bb}{2b}$

or  $b$ , according to the first condition of the problem;

again, the square of the less number  $\frac{a-bb}{2b}$  is

$aa -$

$\frac{aa - 2abb + b^4}{4bb}$ , and the square of the greater  $\frac{a + b}{2}$  is  $\frac{aa + 2ab + b^4}{4bb}$ ; subtract the square of the less from that of the greater, and you will have the difference of their squares  $= \frac{4abb}{4bb} = a$ , as the second condition requires.

### PROBLEM 4.

138. Let  $r$  and  $s$  be two given multipliers, whereof  $r$  is the greater; it is required to divide a given number as  $a$  into two such parts, that the greater part when multiplied into the less multiplier may be equal to the less part when multiplied by the greater multiplier.

### SOLUTION.

Put  $x$  for the greater part, and  $a - x$  for the less; then will the greater part multiplied into the less multiplier be  $sx$ , and the less part multiplied into the greater multiplier will be  $ar - rx$ : but according to the problem, these products are to be equal; therefore  $sx = ar - rx$ , and  $rx + sx = ar$ , but  $rx + sx$  is  $x \times r + s$ ; therefore  $x \times r + s = ar$ ; and  $x$  (the greater of the two parts sought)  $= \frac{ar}{r + s}$ ; whence  $a - x$  (the less part) equal  $\frac{a}{1} - \frac{ar}{r + s} = \frac{ar + as - ar}{r + s} = \frac{as}{r + s}$ ; so the greater part sought is  $\frac{ar}{r + s}$ , and the less  $\frac{as}{r + s}$ .

### THE APPLICATION.

To apply this canon, let it be required to divide 84 into two such parts, that five times one part may be

be equal to seven times the other: here  $a=84$ ,  $r$  the greater multiplier  $=7$ ,  $s=5$ ,  $\frac{ar}{r+s} = \frac{7 \times 84}{12} = 49$ ,

$\frac{as}{r+s} = \frac{5 \times 84}{12} = 35$ ; therefore the greater part is 49, and the less 35; and they will answer the conditions; for first,  $49+35=84$ ; and secondly,  $49 \times 5 = 245 = 35 \times 7$ . Again, let it be required to divide 99 into two such parts, that  $\frac{2}{3}$  of one part may be equal to  $\frac{4}{5}$  of the other: here  $a=99$ ,  $r=\frac{4}{5}$ ,  $s=\frac{2}{3}$ ,  $r+s=\frac{22}{15}$ ,

$\frac{r}{r+s} = \frac{\frac{4}{5}}{\frac{22}{15}} = \frac{6}{11}$ ,  $\frac{s}{r+s} = \frac{\frac{2}{3}}{\frac{22}{15}} = \frac{5}{11}$ ,  $\frac{ar}{r+s} = 99 \times \frac{6}{11}$

$= 54$ ,  $\frac{as}{r+s} = 99 \times \frac{5}{11} = 45$ ; so the two parts are 54 and 45; which is true; for first,  $54+45=99$ ; and secondly,  $\frac{2}{3}$  of 54  $= 36 = \frac{4}{5}$  of 45.

As to the demonstration of this general solution, it must be observed that in this problem there are two conditions; first, that the two parts, when added together, must make  $a$ ; and secondly, that the greater part multiplied into the less multiplier must be equal to the less part multiplied into the greater multiplier: as to the first of the conditions, it is

certain that the parts  $\frac{ar}{r+s}$  and  $\frac{as}{r+s}$  when added

together will make  $\frac{ar+as}{r+s}$ ; but  $ar+as=a \times r+s$ ,

therefore  $\frac{ar+as}{r+s} = a \times \frac{r+s}{r+s} = a \times 1 = a$ : as to the se-

cond condition, if the greater part  $\frac{ar}{r+s}$  be multiplied into  $s$ , the less multiplier, the product will

be  $\frac{ars}{r+s}$ ; and again, if the less part  $\frac{as}{r+s}$  be multiplied into  $r$ , the greater multiplier, the product

will

will also be  $\frac{ars}{r+s}$ ; therefore the two products are equal, as the problem requires; and so the conditions are both satisfied. Q. E. D.

N. B. If any one has a mind to throw the foregoing theorem into words, it may easily be done, and in such a manner as almost to carry its own evidence along with it; for by the rule of proportion,  $r+s$  is to  $r$  as  $a$  to  $\frac{ar}{r+s}$ ; and  $r+s$  is to  $s$  as  $a$  to  $\frac{as}{r+s}$ ; therefore, *As the sum of the two multipliers is to the greater or less multiplier, so is the sum of the two parts sought to the greater or less part*: and this, I say, is pretty evident; for had  $r+s$  been the number to be divided, the parts would certainly have been  $r$  and  $s$ ; therefore if a greater or less number than  $r+s$  is to be divided, the parts ought to be greater or less than  $r$  and  $s$  in the same proportion.

### PROBLEM 5.

139. Let  $r$  and  $s$  be two given multipliers, whereof  $r$  is the greater; it is required to divide a given number  $a$  into two such parts, that  $r$  times one part being added to  $s$  times the other may make some other given number, as  $b$ .

### SOLUTION.

Put  $x$  for the part that is to be multiplied by  $r$ , and consequently  $a-x$  for the other part that is to be multiplied by  $s$ , and the products will be  $rx$  and  $as-sx$ , and their sum will be  $rx+as-sx=b$ ; therefore  $rx-sx=b-as$ , that is,  $x \times r-s=b-as$ ; therefore  $x$  (the part to be multiplied by  $r$ )  $= \frac{b-as}{r-s}$ ; therefore  $a-x$  (the part to be multiplied by  $s$ )  $= \frac{a}{1} - \frac{b-as}{r-s} = \frac{ar-as-b+as}{r-s} = \frac{ar-b}{r-s}$ .

Q

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## THE APPLICATION.

Let it be required to divide 20 into two such parts, that three times one part being added to five times the other may make 84: here  $a=20$ ,  $b=84$ ,  $r=5$ ,  $s=3$ ,  $as=60$ ,  $b-as=24$ ,  $\frac{b-as}{r-s}$  (or the part to be multiplied by 5)  $=\frac{24}{2}=12$ ,  $ar=100$ ,  $ar-b=16$ ;  $\frac{ar-b}{r-s}$  (or the part to be multiplied by 3)  $=\frac{16}{2}=8$ ; therefore the parts sought are 8 and 12; for first,  $8+12=20$ ; and secondly, three times 8 + five times 12 = 84.

Again, let it be required to divide 100 into two such parts, that  $\frac{3}{4}$  of one part being subtracted from  $\frac{5}{6}$  of the other, may leave 39: here it must be observed, that to subtract  $\frac{3}{4}$  of any one quantity from another, is the same as to add  $\frac{-3}{4}$  of it; therefore this problem when reduced to the form of the general one, will stand thus: *To divide a hundred into two such parts, that  $\frac{-3}{4}$  of one part being added to  $+\frac{5}{6}$  of the other may make thirty-nine.* Here  $a=100$ ,  $b=39$ ,  $r=\frac{5}{6}$ ,  $s=\frac{-3}{4}$ ,  $r-s=\frac{5}{6}+\frac{3}{4}=\frac{19}{12}$ ,  $as=\frac{-300}{4}=-75$ ,  $b-as=39+75=114$ ,  $\frac{b-as}{r-s}=\frac{114}{\frac{19}{12}}=72$ ,  $ar=\frac{500}{6}=\frac{250}{3}$ ,  $ar-b=\frac{250}{3}-\frac{39}{1}=\frac{133}{3}$ ,  $\frac{ar-b}{r-s}=\frac{\frac{133}{3}}{\frac{19}{12}}=28$ ; so the two parts are 28 and 72: for  $28+72=100$ ; and moreover  $\frac{3}{4}$  of 28, that

that is, 21, subtracted from  $\frac{5}{6}$  of 72, that is, from 60, leaves 39.

### THE GENERAL DEMONSTRATION.

The two parts  $\frac{ar-b}{r-s}$  and  $\frac{b-as}{r-s}$  when added together, make  $\frac{ar-b+b-as}{r-s} = \frac{ar-as}{r-s} = a \times \frac{r-s}{r-s} = a$ :

again, the part  $\frac{b-as}{r-s}$  being multiplied into  $r$ , its proper multiplier, gives  $\frac{br-ars}{r-s}$ , and the other

part  $\frac{ar-b}{r-s}$ , multiplied into the other multiplier  $s$ , gives  $\frac{ars-bs}{a-s}$ ; add these two products together,

and they will make  $\frac{br-ars+ars-bs}{r-s} = \frac{br-bs}{r-s} = b$ .

Q. E. D.

If any one hereafter shall think me too concise in the solution of these general problems, he must have recourse to the particular ones in the articles I shall refer him to, which he will find explained more at large: and as to the application of these general solutions to those particular cases, it is to be presumed that by this time the learner will be able in some measure to perform that part himself; and therefore I shall for the future leave it to him, except where I shall think my assistance may be of any use.

### PROBLEM 6. (See art. 35.)

140. One meeting a company of beggars, gives to each pence, and has a pence over; but if he would have given them q pence apiece, he would have found he had wanted b pence for that purpose: What was the number of persons?

Q 2

SOLUTION.

## SOLUTION.

The number of persons,  $x$ .

Pence given,  $px$ .

Pence in all,  $px + a$ .

The pence that would have been given upon the other supposition,  $qx$ .

Another expression for the number of pence in all,  $qx - b$ .

Equ.  $qx - b = px + a$ ; therefore  $qx - px - b = a$ ; therefore  $qx - px = a + b$ ; therefore  $x$  (the number of persons)  $= \frac{a+b}{q-p}$ .

## DEMONSTRATION.

If the number of persons be  $\frac{a+b}{q-p}$ , then the pence given will be  $\frac{ap+bp}{q-p}$ , and the pence in all will be  $\frac{ap+bp}{q-p} + \frac{a}{1} = \frac{ap+bp+aq-ap}{q-p} = \frac{aq+bp}{q-p}$ ; again, the number of pence that would have been given upon the second supposition is  $\frac{aq+bq}{q-p}$ ; and therefore the other expression for the number of pence in all will be  $\frac{aq+bq}{q-p} - \frac{b}{1} = \frac{aq+bp}{q-p}$ ; and the perfect agreement between this account and the former is an infallible argument that the number of persons was rightly assigned.

## PROBLEM 7. (See art. 64.)

141. It is required to divide a given number as  $a$  into two such parts, that one part may be to the other as  $r$  to  $s$ .

SOLUTION.

SOLUTION.

The two parts sought,  $x$  and  $a - x$ .

Proportion,  $x$  is to  $a - x$  as  $r$  to  $s$ .

Equation,  $sx = ar - rx$ ; therefore  $rx + sx = ar$ ;

therefore  $x$  (or the first number)  $= \frac{ar}{r+s}$ ; therefore

$a - x$  (or the second number)  $= \frac{a}{1} - \frac{ar}{r+s} = \frac{as}{r+s}$ ;

therefore the two numbers are  $\frac{ar}{r+s}$  and  $\frac{as}{r+s}$ .

DEMONSTRATION.

1st, The two numbers  $\frac{ar}{r+s}$  and  $\frac{as}{r+s}$  when added together make  $\frac{ar+as}{r+s} = a$ .

2dly, The first number  $\frac{ar}{r+s}$  is to the second number  $\frac{as}{r+s}$  as  $ar$  is to  $as$ ; because throwing away the common denominator is no more in reality than multiplying both fractions by it; and every one knows, that the multiplication of two quantities by the same number, makes no alteration in the proportion they bore one to the other: again,  $ar$  is to  $as$  (dividing both by  $a$ ) as  $r$  to  $s$ ; for it is well known that a common division affects proportion no more than a common multiplication: since then the first number is to the second as  $ar$  to  $as$ , and  $ar$  is to  $as$  as  $r$  to  $s$ , it follows, that the first number is to the second as  $r$  to  $s$ . Q. E. D.

PROBLEM 8. (See art. 66.)

142. What number is that, which being severally added to two given numbers,  $a$  a greater number, and  $b$  a less, will make the former sum to the latter as  $r$  to  $s$ ? therefore  $r$  must be greater than  $s$ ?

Q 3

SOLUTION.

## SOLUTION.

The number sought,  $x$ .

Proportion,  $a + x$  is to  $b + x$  as  $r$  to  $s$ .

Equation,  $br + rx = as + sx$ ; therefore  $br + rx - sx = as$ ; therefore  $rx - sx = as - br$ ; therefore  $x = \frac{as - br}{r - s}$ .

## DEMONSTRATION.

The number  $\frac{as - br}{r - s}$  being added to  $a$ , gives  $\frac{ar - br}{r - s}$  and the same number being added to  $b$ , gives  $\frac{as - bs}{r - s}$ ; now  $\frac{ar - br}{r - s}$  is to  $\frac{as - bs}{r - s}$  as  $ar - br$  is to  $as - bs$ , that is, as  $r \times a - b$  is to  $s \times a - b$ , that is, as  $r$  to  $s$ .  
Q. E. D.

## SCHOLIUM.

This problem was to find a number, which, being severally added to  $a$  and  $b$ , will make the former sum to the latter as  $r$  to  $s$ ; let us now change the numbers  $a$  and  $b$  one for another, as also the numbers  $r$  and  $s$  one for another, and then the problem will stand thus: *To find a number, which, being severally added to  $b$  and  $a$ , will make the former sum to the latter as  $s$  to  $r$* : but the condition of this problem is exactly the same with that of the former, and therefore the answer ought still to be the same; that is, as changing  $a$  and  $b$  one for another, and  $r$  and  $s$  one for another, had no effect upon the problem, but left it entirely the same as at first; so if the expression of the number sought be just, the changing of  $a$  and  $b$  one for another, and of  $r$  and  $s$  one for another, ought to make no alteration in that expression, and the number sought ought still to be the same; for truth will always be consistent with herself. Let us try this however, and see what will be the effect of such a

change:

change: now the number sought was  $\frac{as-br}{r-s}$ ; but upon this change,  $as$  becomes  $br$ , and  $br$  becomes  $as$ , and  $r-s$  becomes  $s-r$ , and the whole expression will be turned into this,  $\frac{br-as}{s-r}$ ; but  $\frac{br-as}{s-r}$  is the same as  $\frac{as-br}{r-s}$ ; for changing the sign of both the numerator and denominator of any fraction, no more affects the value of that fraction, than in division the changing of the sign both of the divisor and dividend affects the value of the quotient: thus then we find, that the changing of  $a$  and  $b$  one for another, and of  $r$  and  $s$  one for another, no more affects the theorem for determining the number sought, than it did the problem from whence it was derived.

### PROBLEM 9.

143. It is required to divide a given number as  $a$  into two such parts, that the excess of one part above another given number as  $b$ , may be to what the other wants of  $b$ , as  $r$  to  $s$ ; supposing  $r$  greater than  $s$ .

### SOLUTION.

Put  $x$  for the greater part, and  $a-x$  for the less; then the excess of  $x$  above  $b$  will be  $x-b$ ; and the excess of  $b$  above  $a-x$  will be  $x-a+b$ , as appears by subtracting  $a-x$  from  $b$ ; but by the problem, the former excess is to the latter as  $r$  to  $s$ ; therefore  $x-b$  is to  $x-a+b$  as  $r$  to  $s$ ; multiply extremes and means, and you will have  $sx-bs=rx-ar+br$ ; therefore  $rx-sx=ar-br-bs$ , and  $x$  (the greater part) =  $\frac{ar-br-bs}{r-s}$ ; therefore  $a-x$  (the less part) =  $\frac{a-ar+br-bs}{r-s} = \frac{br+bs-as}{r-s}$ ; so the greater part is

$$\frac{ar - br - bs}{r - s}, \text{ and the less part } \frac{br + bs - as}{r - s},$$

## EXAMPLE.

Let it be required (as in art. 41,) to divide the number 48 into two such parts, that one part may be three times as much above 20 as the other wants of 20: here  $a=48$ ,  $b=20$ ,  $r=3$ ,  $s=1$ ; for to say that the excess must be three times the defect, is no other than to say, that the excess must be to the defect as 3 to 1; the rest is easy.

## THE GENERAL DEMONSTRATION.

1st, The greater part  $\frac{ar - br - bs}{r - s}$ , and the less part  $\frac{br + bs - as}{r - s}$  being added together make  $\frac{ar - as}{r - s} = a$ : again, the excess of the greater part above  $b$ , is  $\frac{ar - br - bs}{r - s} - \frac{b}{1} = \frac{ar - br - bs - br + bs}{r - s} = \frac{ar - 2br}{r - s}$ , and the excess of  $b$  above the less part, which is what the less part wants of  $b$ , is  $\frac{b}{1} - \frac{br - bs + as}{r - s} = \frac{br - bs - br - bs + as}{r - s} = \frac{as - 2bs}{r - s}$ ; therefore the excess of one part above  $b$  is to what the other wants of  $b$ , as  $\frac{ar - 2br}{r - s}$  is to  $\frac{as - 2bs}{r - s}$ , that is, as  $ar - 2br$  is to  $as - 2bs$ , that is, as  $r \times a - 2b$  is to  $s \times a - 2b$ , or as  $r$  to  $s$ . Q. E. D.

## PROBLEM 10. (See art. 55.)

144. *There are two places, whose distance from each other is  $a$ , and from whence two persons set out at the same time with a design to meet, one travelling at the rate of  $p$  miles in  $q$  hours, and the other at the rate of  $r$  miles in  $s$  hours: I demand how long and how far each travelled before they met?*

SOLUTION.

## SOLUTION.

The number of hours travelled by each,  $x$ .

Miles travelled by the first,  $\frac{px}{q}$ .

By the second,  $\frac{rx}{s}$ .

By them both,  $\frac{px}{q} + \frac{rx}{s}$ .

Equation,  $\frac{px}{q} + \frac{rx}{s} = a$ ; therefore  $px + \frac{qrx}{s} = aq$ ;

therefore  $psx + qrx = aqs$ ; therefore  $x$  (or the number of hours travelled by each)  $= \frac{aqs}{ps + qr}$ : now to find

how many miles the first travelled, say, if in  $q$  hours he travelled  $p$  miles, how many will he travel in a number of hours equal to  $\frac{aqs}{ps + qr}$ ? for a fourth num-

ber, I multiply the third number  $\frac{aqs}{ps + qr}$  by the second  $p$ , and the product is  $\frac{apqs}{ps + qr}$ ; this again I di-

vide by the first number  $q$ , and the quotient is  $\frac{aps}{ps + qr}$ ;

for dividing the numerator divides the whole fraction; by the same way of reasoning, the number of miles

travelled by the other will be found to be  $\frac{aqr}{ps + qr}$ ;

therefore the whole number of miles travelled by them

both is  $\frac{aps + aqr}{ps + qr} = a$ , which demonstrates the solution.

## EXAMPLE.

Let the distance of the two places be 154 miles;  
let the first travel at the rate of 3 miles in 2 hours,  
and

and the second after the rate of 5 miles in 4 hours; then we shall have  $a=154$ ,  $p=3$ ,  $q=2$ ,  $r=5$ ,  $s=4$ ,  $ps=12$ ,  $qr=10$ ,  $ps+qr=22$ ,  $\frac{aqs}{ps+qr} = \frac{154 \times 2 \times 4}{22} = 56$ ,  $\frac{aps}{ps+qr} = \frac{154 \times 3 \times 4}{22} = 84$ ,  $\frac{agr}{ps+qr} = \frac{154 \times 2 \times 5}{22} = 70$ : therefore each travelled 56 hours; the first travelled 84 miles, and the other 70.

## S C H O L I U M.

If in the foregoing problem we change  $p$  into  $r$  and  $q$  into  $s$ , and *vice versa*, the consequence will be, that the first traveller will now travel at the same rate as the second did before, and the second at the same rate as the first did before; but the motion whereby these two travellers approach towards each other will still be the same, and therefore the time this motion is performed in, that is, the time that each travelled, must still be the same: let us then make the changes above-mentioned, first in the expression of the time, and see whether that expression will still continue the same; then let us make the same changes in the two expressions of the miles, and see whether by this means these expressions will not be converted each into the other: first then, the expression of the time, which is

$\frac{aqs}{ps+qr}$ , by changing  $p$  into  $r$ , and  $q$  into  $s$ , and *vice*

*versa*, becomes  $\frac{asq}{rq+sp}$ , which is the same as  $\frac{aqs}{ps+qr}$ ,

therefore the expression of the time suffers no alteration by these changes; secondly, the number of miles

travelled by the first was  $\frac{aps}{ps+qr}$ , which, after the

changes abovementioned, becomes  $\frac{arq}{rq+sp}$ , which is

the same as  $\frac{aqr}{ps+qr}$ , the miles travelled by the second;

and

and therefore, *è converso*, the expression  $\frac{aqr}{ps+qr}$  will be changed into the expression  $\frac{aps}{ps+qr}$ ; and thus will the case of the first traveller be changed into that of the second, and *vice versa*.

PROBLEM 15. (See art. 38.)

149. What two numbers are those, whereof the greater is to the less as  $p$  to  $q$ , and the product of their multiplication is to their sum as  $r$  to  $s$ ?

SOLUTION.

Put  $x$  for the less number, and the greater will be found by saying, as  $q$  is to  $p$ , so is  $x$  the less number to  $\frac{px}{q}$  the greater; whence their sum will be  $\frac{px}{q} + \frac{x}{1} = \frac{px+qx}{q}$ : on the other hand, if the greater number  $\frac{px}{q}$  be multiplied into  $x$ , the product will be  $\frac{pxx}{q}$ ; therefore the product of these two numbers will be to their sum as  $\frac{pxx}{q}$  is to  $\frac{px+qx}{q}$ , that is, as  $px$  to  $p+q$ ; but according to the problem, the product is to the sum as  $r$  to  $s$ ; therefore  $px$  is to  $p+q$  as  $r$  to  $s$ ; whence we have this equation,  $psx = pr+qr$ ; and  $x$  (the less number sought)  $= \frac{pr+qr}{ps}$ ; therefore  $px = \frac{pr+qr}{s}$ ; for dividing the denominator multiplies the whole fraction; therefore  $\frac{px}{q}$  (or the greater number)  $= \frac{pr+qr}{qs}$ .

DEMON-

## DEMONSTRATION.

1st, The greater number is to the less as  $\frac{pr+qr}{qs}$  is to  $\frac{pr+qr}{ps}$ ; divide  $pr+qr$  by itself, and the quotient will be 1; so that we may now say, that the greater number is to the less as  $\frac{1}{qs}$  is to  $\frac{1}{ps}$ , that is, as  $\frac{1}{q}$  is to  $\frac{1}{p}$ , that is, as  $\frac{p}{q}$  is to 1, that is, as  $p$  is to  $q$ .

2dly, The greater number  $\frac{pr+qr}{qs}$  and the less  $\frac{pr+qr}{ps}$  being added together make  $\frac{pprs + pqrs + pqrs + qqrs}{pqss}$   
 $= \frac{pprs + 2pqrs + qqrs}{pqss}$ ; but  $pp + 2pq + qq = p + q$ ; therefore the sum of the two numbers sought is  $\frac{rs \times p + q}{pqss}$ .

3dly, The greater number  $\frac{pr+qr}{qs}$  multiplied into the less  $\frac{pr+qr}{ps}$  produces  $\frac{rr \times p + q}{pqss}$ .

4thly, Therefore the product of the two numbers sought is to their sum as  $\frac{rr \times p + q}{pqss}$  is to  $\frac{rs \times p + q}{pqss}$ , that is, as  $rr$  is to  $rs$ , or as  $r$  to  $s$ . Q. E. D.

## PROBLEM 17.

151. What two numbers are those, the product of whose multiplication is  $p$ , and the quotient of the greater divided by the less is  $q$ ?

SOLUTION.

SOLUTION.

Put  $x$  for the greater number, and consequently  $\frac{p}{x}$  for the less; then will the quotient of the greater divided by the less be  $\frac{xx}{p}$ ; but, according to the problem, this quotient ought to be  $q$ ; therefore  $\frac{xx}{p} = q$ ; and  $xx = pq$ , and  $x$  (the greater number sought)  $= \sqrt{pq}$ : again, since  $xx = pq$ , we have  $\frac{pp}{xx} = \frac{pp}{pq} = \frac{p}{q}$ ; and  $\frac{p}{x}$  (or the less number sought)  $= \sqrt{\frac{p}{q}}$ ; so that the greater of the two numbers sought is  $\sqrt{pq}$ , and the less  $\sqrt{\frac{p}{q}}$ .

EXAMPLE.

Let the product of the two numbers sought be 144, and the quotient of the greater divided by the less 16; then we shall have  $p = 144$ ,  $q = 16$ ,  $pq = 144 \times 16$ ,  $\sqrt{pq} = 12 \times 4 = 48$ ;  $\frac{p}{q} = \frac{144}{16}$ ,  $\sqrt{\frac{p}{q}} = \frac{12}{4} = 3$ ; therefore the numbers are 48 and 3.

DEMONSTRATION.

1st,  $pq$  multiplied into  $\frac{p}{q}$  gives  $\frac{ppq}{q} = pp$ ; therefore  $\sqrt{pq}$  multiplied into  $\sqrt{\frac{p}{q}}$  gives  $p$ .

2dly,  $pq$  being divided by  $\frac{p}{q}$  gives  $\frac{pq q}{p} = qq$ ; therefore  $\sqrt{pq}$  being divided by  $\sqrt{\frac{p}{q}}$  gives  $q$ . Q. E. D.

PROBLEM

## PROBLEM 21. (See art. 130.)

155. *What two numbers are those, whose difference being multiplied into the difference of their squares will make a, and whose sum being multiplied into the sum of their squares will make b?*

## SOLUTION.

For the two numbers sought put  $x$  and  $y$ ; then according to the first supposition,  $x - y \times x^2 - y^2$ , or  $x - y \times x - y \times x + y$ , or  $xx - 2xy + y^2 \times x + y = a$ ; therefore

$$\text{Equ. 1st, } x^2 - 2xy + y^2 = \frac{a}{x + y}.$$

Again, according to the second supposition,  $x + y \times x^2 + y^2 = b$ ; therefore

$$\text{Equ. 2d, } x^2 + y^2 = \frac{b}{x + y}.$$

From twice the second equation subtract the first,

$$\text{that is, from } 2x^2 + 2y^2 = \frac{2b}{x + y}$$

$$\text{subtract } x^2 - 2xy + y^2 = \frac{a}{x + y},$$

$$\text{and there will remain } x^2 + 2xy + y^2 = \frac{2b - a}{x + y},$$

$$\text{that is, } x + y = \frac{2b - a}{x + y}; \text{ therefore } x + y = \sqrt[3]{2b - a}; \text{ make}$$

$2b - a = r^3$ , that is, put  $r$  for the cube root of  $2b - a$ , and you will have

$$\text{Equ. 3d, } x + y = r.$$

Again, in the first equation we had  $x^2 - 2xy + y^2 =$

$$\frac{a}{x + y} = \frac{a}{r}, \text{ that is, } x - y = \frac{a}{r}; \text{ make } \frac{a}{r} = ss, \text{ that}$$

is,

is, put  $s$  for the square root of  $\frac{a}{r}$ , and you will have

Equ. 4th,  $x - y = s$ .

Add the third and fourth equations together, and you will have  $2x = r + s$ , and  $x = \frac{r+s}{2}$ ; subtract the fourth equation from the third, and you will have  $2y = r - s$ , and  $y = \frac{r-s}{2}$ ; whence we have the following canon:

Make  $2b - a = r^3$ , and  $\frac{a}{r} = s^2$ , and the numbers sought will be  $\frac{r+s}{2}$ , and  $\frac{r-s}{2}$ .

#### DEMONSTRATION.

The difference of the numbers  $\frac{r+s}{2}$  and  $\frac{r-s}{2}$  is  $s$ , and the difference of their squares is  $rs$ , as is easily tried; therefore the difference of the numbers multiplied into the difference of their squares is  $rss = \frac{ra}{r} = a$ : again, the sum of the numbers  $\frac{r+s}{2}$  and  $\frac{r-s}{2}$  is  $r$ , and the sum of their squares is  $\frac{r^2 + s^2}{2}$ ; therefore the sum of the numbers multiplied into the sum of their squares is  $\frac{r^3 + rss}{2}$ ; but  $r^3 = 2b - a$  by the canon, and  $rss = a$  by the same; therefore the sum of the numbers multiplied into the sum of their squares is  $\frac{2b - a + a}{2} = b$ . Q. E. D.

PROBLEM

## PROBLEM 22.

156. Out of a common pack of fifty-two cards, let part be distributed into several distinct parcels or heaps in the manner following: upon the lowest card of every heap let as many others be laid as are sufficient to make up its number twelve; as, if four be the number of the lowest card, let eight others be laid upon it; if five, let seven; if  $a$ , let twelve  $- a$ , &c.: It is required, having given the number of heaps, which we shall call  $n$ , as also the number of cards still remaining in the dealer's hand, which we shall call  $r$ , to find the sum of the numbers of all the bottom cards put together.

## SOLUTION.

Let  $a, b, c$ , &c. express the number of the bottom card in the several heaps: then will  $12 - a$  express the number of all the cards lying upon the bottom card of the first heap, that is, the number of all the cards of the first heap except the lowest, will be  $12 - a$ ; therefore  $13 - a$  will be the number of all the cards in the first heap; for the same reason,  $13 - b$  will be the number of all the cards in the second heap; and  $13 - c$  the number of all those in the third, and so on; therefore the number of all the cards in all the heaps will be  $13 \times n - a - b - c$ , &c.: make  $a + b + c$ , &c. (or the sum of the number of all the bottom cards)  $= x$ , and then we shall have the numbers of all the cards drawn out into heaps  $= 13n - x$ ; but these, together with  $r$ , the number of cards undrawn out, make up the whole pack 52; therefore we have this equation,  $13n - x + r = 52$ ; therefore  $x + 52 = 13n + r$ ; therefore  $x = 13n - 52 + r$ ; but  $52 = 13 \times 4$ ; therefore  $13n - 52 = 13 \times n - 4$ ; therefore  $x = 13 \times n - 4 + r$ ; in words thus: From the number of heaps subtract four; multiply the rest by thirteen; and this product, added to the number of cards still remaining in the dealer's hand, will

Art. 157, 158. and Theorems deduced from them. 257

will give the sum of the numbers of all the bottom cards put together: as for example, let there be three heaps, and thirty cards remaining; now 4 subtracted from 3 leaves  $-1$ ; this multiplied by 13 gives  $-13$ , and this product added to 30, the number of cards remaining, gives 17 for the sum of the numbers of all the bottom cards.

A more universal theorem is as follows:

Let  $n$  be the number of heaps as before,  $p$  the number of cards in a pack; let as many cards be laid upon the lowest of every heap as are sufficient to make up its number  $q$ ; and lastly, let  $r$  be the number of remaining cards as before; and the sum of the numbers of all the bottom cards will be found to be  $q + 1 \times n + r - p$ .

PROBLEM 24. (See art. 111.)

158. What two numbers are those, whose sum is  $a$ , and the product of whose multiplication is  $b$ ?

SOLUTION.

The two numbers sought,  $x$  and  $a - x$ .

The product of their multiplication,  $ax - xx = b$ ; whence, changing the signs,  $xx - ax = -b$ , and com-

pleting the square,  $xx - ax + \frac{aa}{4} = \frac{aa}{4} - b = \frac{aa - 4b}{4}$

$= \frac{s s}{4}$ ; extract the square root of both sides, that is,

of  $x^2 - ax + \frac{aa}{4}$  on one side, and of  $\frac{s s}{4}$  on the other,

and you will have  $x - \frac{a}{2} = \pm \frac{s}{2}$ , and  $x = \frac{a \pm s}{2}$ ;

whence the following canon:

Make  $aa - 4b = ss$ , and the greater number will be  $\frac{a+s}{2}$ , and the less number  $\frac{a-s}{2}$ .

## The SYNTHETICAL DEMONSTRATION.

1st,  $\frac{a+s}{2}$  added to  $\frac{a-s}{2}$  gives  $\frac{2a}{2}$  or  $a$ .

2dly,  $\frac{a+s}{2}$  multiplied into  $\frac{a-s}{2}$  gives  $\frac{aa-ss}{4} =$

(by substituting  $-aa+4b$  instead of  $-ss$ )  $\frac{aa-aa+4b}{4}$   
 $= \frac{4b}{4} = b$ . Q. E. D.

*An Example to the foregoing Canon.*

What two numbers are those, whose sum is twenty-five, and the product of whose multiplication is 144?

Here  $a=25$ ,  $b=144$ ,  $aa-4b$  or  $ss=49$ ,  $s=7$ ,  $\frac{a+s}{2}$   
 $=16$ ,  $\frac{a-s}{2}=9$ ; so the numbers are 9 and 16.

## PROBLEM 25. (See art. 113.)

159. What two numbers are those, whose sum is  $a$ , and the sum of their squares  $b$ ?

## SOLUTION.

The two numbers sought,  $x$  and  $a-x$ .

The square of the former,  $xx$ .

The square of the latter,  $aa-2ax+xx$ .

The sum of their squares  $aa-2ax+2xx=b$ ; therefore  $2xx-2ax=b-aa$ , and  $xx-ax=\frac{b-aa}{2}$ , and

$xx-ax+\frac{aa}{4}=\frac{aa}{4}+\frac{b-aa}{2}=\frac{2b-aa}{4}=\frac{ss}{4}$ ; extract

the square roots, that is, the root of  $xx-ax+\frac{aa}{4}$

on one side, and of  $\frac{ss}{4}$  on the other, and you will have  $x - \frac{a}{2} = \pm \frac{s}{2}$ , and  $x \pm \frac{a \pm s}{2}$ ; whence the following canon :

Make  $2b - aa = ss$ , and you will have  $\frac{a+s}{2}$  for the greater number, and  $\frac{a-s}{2}$  for the less.

### DEMONSTRATION

1st,  $\frac{a+s}{2}$  added to  $\frac{a-s}{2}$  gives  $a$ .

2dly, The square of  $\frac{a+s}{2}$  is  $\frac{aa + 2as + ss}{4}$ ; the square of  $\frac{a-s}{2}$  is  $\frac{aa - 2as + ss}{4}$ ; and therefore the sum of their squares is  $\frac{2aa + 2ss}{4} = \frac{aa + ss}{2} =$  (by the canon)  $\frac{aa + 2b - aa}{2} = b$ . Q. E. D

*An Example to the foregoing Canon.*

What two numbers are those, whose sum is 28, and the sum of their squares 400? Here  $a = 28$ ,  $b = 400$ ,  $2b - aa$  or  $ss = 16$ ,  $s = 4$ .  $\frac{a+s}{2} = 16$ ,  $\frac{a-s}{2} = 12$ ; therefore the numbers are 12 and 16.

PROBLEM 26. (See art. 114.)

160. What two numbers are those, whose sum is  $a$ , and the sum of their cubes  $b$ ?

### SOLUTION.

The two numbers sought,  $x$  and  $a - x$ .

The cube of the former,  $x^3$ .

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The cube of the latter,  $a^3 - 3a^2x + 3ax^2 - x^3$ .

The sum of their cubes,  $a^3 - 3a^2x + 3ax^2 = b$ ; therefore  $3ax^2 - 3a^2x = b - a^3$ ; divide by  $3a$ , and you will have  $xx - ax = \frac{b - a^3}{3a}$ , and  $xx - ax + \frac{aa}{4} = \frac{aa}{4} + \frac{b - a^3}{3a} = \frac{4b - a^3}{12a} = \frac{1}{4} \times \frac{4b - a^3}{3a} = \frac{ss}{4}$ ; extract the square root of both sides, that is, of  $xx - ax + \frac{aa}{4}$  on one side, and of  $\frac{ss}{4}$  on the other, and you will have  $x - \frac{a}{2} = \pm \frac{s}{2}$  and  $x = \frac{a \pm s}{2}$ ; whence the following canon:

Make  $\frac{4b - a^3}{3a} = ss$ , and you will have  $\frac{a + s}{2}$  for the greater number, and  $\frac{a - s}{2}$  for the less.

#### DEMONSTRATION.

1st,  $\frac{a + s}{2}$  added to  $\frac{a - s}{2}$  gives  $a$ .

2dly, The cube of  $\frac{a + s}{2}$  is  $\frac{a^3 + 3a^2s + 3as^2 + s^3}{8}$ , and the cube of  $\frac{a - s}{2}$  is  $\frac{a^3 - 3a^2s + 3as^2 - s^3}{8}$ ; therefore, the sum of their cubes is  $\frac{2a^3 + 6as^2}{8} = \frac{a^3 + 3ass}{4} = \frac{a^3 + 4b - a^3}{4}$  by the canon,  $= b$ . Q. E. D.

*An Example to the foregoing Canon.*

What two numbers are those, whose sum is 7, and the sum of their cubes 133? Here  $a = 7$ ,  $b = 133$ ,  
 $4b - a^3$

$\frac{4b-a^3}{3a}$  or  $ss=9$ ,  $s=3$ ,  $\frac{a+s}{2}=5$ ,  $\frac{a-s}{2}=2$ ; so the numbers are 5 and 2.

# PROBLEM 27.

161. It is required to find two numbers whose difference is  $d$ , and which, dividing a given number as  $a$ , will have two quantities whose difference is  $b$ .

## SOLUTION.

The two numbers sought,  $x$  and  $x+d$ .

The two quotients  $\frac{a}{x}$  and  $\frac{a}{x+d}$ .

Their difference,  $\frac{a}{x} - \frac{a}{x+d} = \frac{ad}{xx+dx} = b$ ; therefore  $bxx+bdx=ad$ , and  $xx+dx=\frac{ad}{b}$ ; therefore  $xx+dx+\frac{dd}{4} = \frac{ad}{b} + \frac{dd}{4} = \frac{1}{4} \times \frac{4ad}{b} + dd = \frac{ss}{4}$ ; extract the square root of  $xx+dx+\frac{dd}{4}$  on one side, and of  $\frac{ss}{4}$  on the other, and you will have  $x+\frac{d}{2} = \pm \frac{s}{2}$ , whence  $x = \frac{s-d}{2}$  or  $\frac{-s-d}{2}$ ; set aside the negative root, and you will have  $x$  (the less divisor) =  $\frac{s-d}{2}$ , and  $x+d$  (the greater) =  $\frac{s-d}{2} + \frac{d}{1} = \frac{s+d}{2}$ ; and we shall have the following canon:

Make  $\frac{4ad}{b} + dd = ss$ , and you will have  $\frac{s+d}{2}$  for the greater divisor, and  $\frac{s-d}{2}$  for the less.

N. B. That  $\frac{s-d}{2}$  is an affirmative quantity, is evident from hence, that  $ss = \frac{4ad}{b} + dd$ ; therefore  $ss$  is greater than  $dd$ , and  $s$  greater than  $d$ ; therefore  $\frac{s-d}{2}$  is affirmative.

*The Demonstration of the Canon.*

1st, If the less divisor  $\frac{s-d}{2}$  be subtracted from the greater  $\frac{s+d}{2}$ , the remainder will be  $d$ ; therefore the difference of the divisors is  $d$ .

2dly, If the dividend  $a$  be severally divided by the two divisors  $\frac{s-d}{2}$  and  $\frac{s+d}{2}$ , the two quotients will be  $\frac{2a}{s-d}$  and  $\frac{2a}{s+d}$  respectively, whereof the former will be the greater, as having a less denominator; therefore the difference of the quotients is  $\frac{2a}{s-d} - \frac{2a}{s+d} = \frac{2as + 2ad - 2as + 2ad}{ss - dd} = \frac{4ad}{ss - dd} = \frac{4ad}{\frac{4ad}{b}}$  by the canon,  $= b$ . Q. E. D.

*An Example to the foregoing Canon.*

Let it be required to find two divisors whose difference is 1, and which, dividing a given number as 144, will have two quotients whose difference is 2.

Here  $a = 144$ ,  $b = 2$ ,  $d = 1$ ,  $\frac{4ad}{b} + dd$  or  $ss = 289$ ,  
 $s = 17$ ,

$s=17$ ,  $\frac{s+d}{2}=9$ ,  $\frac{s-d}{2}=8$ ; therefore the divisors are 8 and 9, and the quotients 18 and 16.

# SCHOLIUM.

If in this last problem we had put  $x$  for the greater quantity, and  $x-d$  for the less, the equation would have been  $\frac{a}{x-d}-\frac{a}{x}=b$ , or  $\frac{ad}{xx-dx}=b$ , which is different from the former; and therefore it could not be expected that, in that equation, the two roots should be the numbers sought, but rather the two different values of  $x$ , the lesser of them.

## PROBLEM 28. (See art. 118.)

162. *What number is that, which, being added to its square root, will make  $a$ ?*

# SOLUTION.

Put  $xx$  for the number sought, and you will have this equation,  $xx+1x=a$ ; therefore  $xx+1x+\frac{1}{4}=a+\frac{1}{4}=\frac{4a+1}{4}=\frac{ss}{4}$ ; therefore  $x+\frac{1}{2}=\pm\frac{s}{2}$ ; therefore  $x=\frac{s-1}{2}$  or  $\frac{-s-1}{2}$ : If  $x$  be made  $=\frac{s-1}{2}$ , you will have  $xx=\frac{ss-2s+1}{4}$ ; if  $x$  be made equal to  $\frac{-s-1}{2}$ , you will have  $xx=\frac{ss+2s+1}{4}$ ; whence the following canon:

*Make  $4a+1=ss$ , and the number sought will be  $\frac{ss-2s+1}{4}$  or  $\frac{ss+2s+1}{4}$ , according to the square root to be added is taken affirmatively or negatively.*

## DEMONSTRATION.

*Case 1st*, If to the number  $\frac{ss - 2s + 1}{4}$  be added its affirmative square root  $\frac{s - 1}{2}$ , or  $\frac{2s - 2}{4}$ , the sum will be  $\frac{ss - 1}{4} = a$ , by the canon.

*Case 2d*, If to the number  $\frac{ss + 2s + 1}{4}$  be added its negative square root  $\frac{-s - 1}{2}$ , or  $\frac{-2s - 2}{4}$ , the sum will again be  $\frac{ss - 1}{4} = a$ , as before. Q. E. D.

## PROBLEM 31.

165. *What two numbers are those, whose sum added to the sum of their squares is a, and whose difference added to the difference of their squares is b?*

## SOLUTION.

Put  $x$  and  $y$  for the two numbers sought, and the fundamental equations will be 1st,  $x + y + x^2 + y^2 = a$ ; 2dly,  $x - y + x^2 - y^2 = b$ ; which equations when reduced to order will stand thus;

$$\text{Equ. 1st, } xx + x + yy + y = a.$$

$$\text{Equ. 2d, } xx + x - yy - y = b.$$

Add these two last equations together, and you will have  $2xx + 2x = a + b$ ; whence  $xx + 1x = \frac{a+b}{2}$ , and  $xx + 1x + \frac{1}{4} = \frac{a+b}{2} + \frac{1}{4} = \frac{2a + 2b + 1}{4}$

$= \frac{rr}{4}$ ; extract the root of  $xx + 1x + \frac{1}{4}$  on one

side, and of  $\frac{rr}{4}$  on the other, and you will have

$$x + \frac{1}{2} = \frac{r}{2} \text{ and } x = \frac{r-1}{2} : \text{ again, subtract the}$$

second

second equation from the first, and you will have  $2y^2 + 2y = a - b$ ; and  $y^2 + y = \frac{a-b}{2}$ , and  $y^2 + 1y + \frac{1}{4} = \frac{2a - 2b + 1}{4} = \frac{ss}{4}$ ; whence  $y + \frac{1}{2} = \frac{s}{2}$ , and  $y = \frac{s-1}{2}$ ; whence the following canon.

Make  $2a + 2b + 1 = rr$ , and  $2a - 2b + 1 = ss$ , and you will have the greater number equal to  $\frac{r-1}{2}$ , and the less number  $= \frac{s-1}{2}$ .

# DEMONSTRATION.

The sum of  $\frac{r-1}{2}$  and  $\frac{s-1}{2}$  is  $\frac{r+s-2}{2}$ , or  $\frac{2r+2s-4}{4}$ .

The square of  $\frac{r-1}{2}$  is  $\frac{r^2-2r+1}{4}$ .

The square of  $\frac{s-1}{2}$  is  $\frac{s^2-2s+1}{4}$ .

therefore the sum of their square is  $\frac{r^2+s^2-2r-2s+2}{4}$ ;

add to this the sum of the numbers above found, to wit,  $\frac{2r+2s-4}{4}$ , and you will have the sum of the numbers added to the sum of their squares equal to  $\frac{r^2+ss-2}{4}$ ; but  $r^2+s^2=4a+2$  by the canon; there-

fore  $rr+ss-2=4a$ , and  $\frac{r^2+s^2-2}{4}$ , or the sum of the numbers added to the sum of their squares, equals  $a$ : again, the difference of  $\frac{r-1}{2}$  and  $\frac{s-1}{2}$  is

$\frac{r-s}{2}$  or  $\frac{2r-2s}{4}$ ; and the difference of their squares is  $\frac{r^2-s^2-2s-2r}{4}$ ; therefore the difference of the numbers added to the difference of their squares is  $\frac{r^2-s^2-2s-2r}{4} = \frac{4b}{4}$  by the canon,  $=b$ . Q. E. D.

*An Example to the foregoing Canon.*

Let the sum of the numbers added to the sum of their squares be 26, and their difference added to the difference of their squares 14; and we shall have  $a = 26$ ,  $b = 14$ ,  $2a + 2b + 1$  or  $rr = 81$ ,  $r = 9$ ,  $\frac{r-1}{2} = 4$ ,  $2a - 2b + 1$  or  $ss = 25$ ,  $s = 5$ ,  $\frac{s-1}{2} = 2$ ; and so the numbers sought will be 4 and 2.

PROBLEM 32.

166. *What two numbers are those, the sum of whose squares is a, and the product of their multiplication b?*

SOLUTION.

For the two numbers sought put  $x$  and  $\frac{b}{x}$ , and the sum of their squares will be  $x^2 + \frac{b^2}{x^2} = a$ ; therefore  $x^4 + b^2 = ax^2$ ; therefore  $x^4 - ax^2 = -bb$ , and  $x^4 - axx + \frac{aa}{4} = \frac{aa}{4} - bb = \frac{aa - 4bb}{4} = \frac{ss}{4}$ ; extract the square root of  $x^4 - axx + \frac{aa}{4}$  on one side, and of  $\frac{ss}{4}$  on the other, and you will have  $x^2 - \frac{a}{2} = \pm \frac{s}{2}$ , and  $x^2 = \frac{a \pm s}{2}$ ; and since this equation will be the same, which-

whichsoever of the unknown quantities  $x$  is made to stand for, you will have the following canon :

*Make*  $aa - 4bb = ss$ , *and you will have the square of the greater number equal to*  $\frac{a+s}{2}$ , *and the square of the less equal to*  $\frac{a-s}{2}$ .

# D E M O N S T R A T I O N .

If the square of the greater number, which is  $\frac{a+s}{2}$ , be added to the square of the less number, which is  $\frac{a-s}{2}$ , the sum of their squares will be  $\frac{2a}{2}$  or  $a$  : again, if the square of the greater number, which is  $\frac{a+s}{2}$ , be multiplied into the square of the less number, which is  $\frac{a-s}{2}$ , the product of these two squares will be  $\frac{aa - ss}{4} = \frac{aa - aa + 4bb}{4}$  by the canon,  $= \frac{4bb}{4} = bb$  ; but if the square of the greater number multiplied into the square of the less gives  $bb$ , then the greater number multiplied into the less will give  $b$ . Q. E. D.

## An Example to the foregoing Canon.

Let the sum of the squares of the two numbers sought be 400, and the product of their multiplication 192 ; then you will have  $a = 400$ ,  $b = 192$ ,  $a^2 - 4b^2$  or  $s^2 = 12544$ ,  $s = 112$ ,  $\frac{a+s}{2}$  or the square of the greater number  $= 256$ ,  $\frac{a-s}{2}$  or the square of the less number  $= 144$  ; therefore the greater number is 16, and the less 12.

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T H E

E L E M E N T S    o f    A L G E B R A.

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B O O K    V.

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*In what Cases a Problem may admit of many Answers.*

*Art. 168.* **I**T has already been observed, that if in any problem the number of independent conditions be equal to the number of unknown quantities, such a problem will admit but of one solution; or if it admits of more, they will however be so determined as to leave no room for arbitrary positions: but if the conditions be fewer in number than are the unknown quantities, those that are wanting may then be supplied by the Analyst himself at pleasure; and as there is infinite choice, it is no wonder if in such a case a problem admits of an infinite number of answers, especially where fractions are taken into that number; but if the problem relates to whole numbers only, then the number of answers will sometimes be finite and sometimes infinite, as the nature of the problem will bear. This will be sufficiently illustrated by the two following examples:

Ex-

EXAMPLE 1.

*Let it be required to find two numbers, whose sum is equal to ten times their difference.*

Here putting  $x$  and  $y$  for the two numbers sought, it is plain that in this case we have but one condition, and consequently but one equation, to wit,  $x + y = 10x - 10y$ , which equation being reduced, gives  $x = \frac{11y}{9}$ ; and this is all the problem requires.

Here then it is plain that the Analyst is entirely at liberty to substitute whatever whole number, mixt number, or proper fraction, he pleases for  $y$ , provided he does but make  $x = \frac{11y}{9}$ ; and the two quantities

$x$  and  $y$  will solve the problem. As for instance, let  $\frac{1}{2}$  be put for  $y$ ; then will  $x$  or  $\frac{11y}{9}$  be  $\frac{11}{18}$ , and those two fractions  $\frac{1}{2}$  and  $\frac{11}{18}$  will solve the problem; for their difference is  $\frac{1}{9}$ , and their sum  $\frac{10}{9}$ . But if it be intended that  $x$  and  $y$  shall both be whole numbers, then such a whole number must be substituted for  $y$  as will admit of 9 for a divisor without a remainder: but of such whole numbers there is infinite choice, as 9, 18, 27, 36, &c.; therefore this question is capable of an infinite number of answers, both in whole numbers and fractions.

EXAMPLE 2.

*Let it now be required to find two numbers  $x$  and  $y$ , the product of whose multiplication is equal to ten times their difference.*

Here the equation will be  $yx = 10x - 10y$ , which being reduced, gives  $x = \frac{10y}{10 - y}$ . Here it is plain that  $y$  must be less than 10; for if  $y$  was equal to 10, the

the fraction  $\frac{10y}{10-y}$  would be infinite, as will be shewn in another place; and if  $y$  be greater than 10, then  $10-y$ , and consequently  $\frac{10y}{10-y}$  will be a negative quantity, whereas the problem may be supposed to relate to affirmative quantities only: however, as there is infinite choice of fractions between 0 and 10, and as any of these may be substituted for  $y$ , the problem will still be capable of an infinite number of solutions, if fractions may be admitted; but if it be required that  $x$  and  $y$  be both whole numbers, then there cannot be above nine such numbers that can be put for  $y$ ; nor perhaps all these neither, as remains in the next place to be shewn. Now to find what whole number being put for  $y$  will bring out  $x$  a whole number also, I reduce the quantity  $\frac{10y}{10-y}$  to a more simple one, by dividing  $10y$  by  $10-y$ , or rather by  $-y+10$ , beginning with  $-y$  thus:  $10y$  divided by  $-y$  quotes  $-10$ , which I put down in the quotient; then multiplying the divisor  $-y+10$  by  $-10$ , the quotient, I find the product to be  $+10y-100$ , which being subtracted from  $10y$  the dividend, leaves 100 for a remainder; but not intending to carry on the division any farther, I represent the rest of the quotient by the fraction  $\frac{100}{10-y}$ ; so  $x = \frac{100}{10-y} - 10$ ; therefore, that  $x$  may be a whole number, it is necessary that  $\frac{100}{10-y}$  be a whole number; but this will be impossible, unless  $10-y$  be some one of the divisors of 100, I mean such a number as will divide 100 without remainder: I enquire therefore in the next place, how many such divisors 100 will admit of that are under 10; for so long as  $y$  is any thing,  $10-y$  must be less than 10; and I find four such divisors,

Art. 168. *Problems which admit of many Answers.* 271  
 to wit, 1, 2, 4, and 5; therefore if  $10-y$  be put  
 equal to any of these,  $x$  or  $\frac{100}{10-y} - 10$  must come  
 out a whole number; and it must also come out affir-  
 mative; for so long as  $10-y$  is greater than nothing  
 and less than 10,  $\frac{100}{10-y}$  will always be greater than  
 $\frac{100}{10}$ , that is, than 10, and consequently  $\frac{100}{10-y} - 10$   
 or  $x$  will be affirmative. Let us then suppose first,  
 $10-y=1$ , and we shall have  $y=9$ , and  $\frac{100}{10-y}$  or  
 $x=90$ . 2dly, if  $10-y=2$ , we shall have  $y=8$ ,  
 and  $x=40$ . 3dly, if  $10-y=4$ , we shall have  $y=6$ ,  
 and  $x=15$ . Lastly, if  $10-y=5$ , we shall have  
 $y=5$ , and  $x=10$ : therefore this question admits of  
 4 solutions in whole numbers, to wit, 90 and 9,  
 40 and 8, 15 and 6, and 10 and 5; all which equally  
 answer the condition of the problem, as will appear  
 upon trial.

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T H E  
E L E M E N T S   O F   A L G E B R A.

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B O O K   V I I .  
O F   P R O P O R T I O N .

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*Of the necessity of resuming the Doctrine of Proportion, and removing some Difficulties which seem to attend it, as delivered in the Elements.*

*Art. 261.* **I**N the 15th and 16th articles of this treatise I have laid down as clearly, and yet as succinctly, as I was able, the doctrine of proportion so far as it relates to numbers and commensurable quantities, whereof any one may be considered as some multiple, part, or parts, of another of the same kind; and it served well enough all the purposes it was designed for. But being in the next book to apply Algebra to Geometry, and so to consider proportion as it relates to magnitudes in general, whether commensurable or incommensurable, I should come short of the ἀπλῆς γεωμετρικῆς, was I not to resume this subject, and to consider it now in its full extent as it is laid down in the fifth book of the Elements of Geometry.

metry. I might indeed have excused myself from this part of my task, and should have been very glad to have done it, by referring the reader at once to the elements themselves without any further assistance; but I could not withstand some reasons drawn from experience, which to me seemed to plead very powerfully to the contrary.

I frequently observe, that most of those who set themselves to read *Euclid*, when they come at the fifth book, which treats of proportion, either entirely pass it by as containing something too subtil to be comprehended by young beginners, or else touch so very slightly upon it as to be little the better for it; and thus the doctrine of proportion (which is certainly the most extensive, and consequently the most useful, part of the Mathematics) is either taken for granted, or at best but partially understood by them. The schemes there made use of are scarce bold enough, I had almost said, scarce complicated enough, to affect the imagination so strongly as is necessary to fix the attention.

The first, second, third, fifth, and sixth, propositions are self-evident, as well as some others, and upon that very account create an impatient reader much greater uneasiness than if they were farther removed from common sense; because the truths from whence these propositions are deduced are not so distinct from the propositions themselves as in many other cases. But it ought to be considered, that the perfection of all arts and sciences in general, and of Geometry in particular, is, to subsist upon as few first principles or axioms as is possible; and therefore, whenever a proposition, how evident soever it may appear in itself, can be deduced from any that is gone before, it ought by all means to be so deduced, and not to be made a first principle, and so unnecessarily to increase their number.

The design of a geometrical demonstration is not so much to illustrate the proposition to which it is

annexed, or to render it more evident than it would have been without it (though this ought certainly to be done wherever the nature of things will permit), as it is to shew the necessary connection the proposition to be demonstrated has with some previous truth already admitted or proved, so as to stand and fall together, whether such previous truths be more or less evident than the proposition to be demonstrated: I say more or less evident; for it is not uncommon in the course of *Euclid's* geometry to meet with propositions demonstrated from others that are less evident than themselves. For an instance of this we need go no farther than the twentieth proposition of the first book, where it is demonstrated that *in every triangle any two sides taken together are greater than the third*: now it is certain that this proposition is more evident than that the external angle is greater than either of the internal and opposite ones; and yet the former, by the help of the 19th proposition, is demonstrated from the latter.

But there is another reason to be given for demonstrating self-evident propositions in many cases, and particularly in this fifth book of the elements. A proposition may sometimes be taken to be self-evident according to our narrow and scanty notions of things, which, when better understood, will be found to be otherwise. These propositions, to wit, that *equal quantities will have the same proportion to a third*, that *of two unequal quantities the greater will have a greater proportion to a third than the less*, and some others of the same stamp in the fifth book, are such as will pass with most for self-evident propositions; and so they are without all doubt according to the common conception of proportionality; but when they come to be examined according to the juster and more extensive idea *Euclid* has given of it, I fear they will both, and the latter more especially, be found to want demonstration.

In a perfect and regular system of elementary Geometry, such a one as that of *Euclid* may be supposed to be, or at least to have been, certain properties of lines, angles, and figures, are to be laid down, and those of the simplest kind, for definitions; from whence, and from one another, all the rest are to be derived with the utmost rigour, without the least appeal even to common sense. Common sense is by no means to be made the standard of any geometrical truths whatever, except first principles: its province must be only to judge whether a proposition be duly demonstrated according to the rules already prescribed, that is, whether the necessary connection it has with any previous truth be clearly and distinctly made out; when that is done, nothing remains but to pass sentence. Whilst the science continues thus circumscribed, no mistakes, no disputes, can arise concerning its boundaries; but whenever these come to be transgressed, such a loose will be given to Geometry that it would be impossible to agree upon any others whereby to restrain it.

Thus much I thought proper to lay down concerning the nature of a geometrical demonstration, that young students may not sometimes think themselves disappointed, or not proceed with that coolness and judgement absolutely necessary to conduct them through the elements of Geometry.

But as to the matter in hand, there is another difficulty still behind, which I believe is often a greater discouragement to young beginners in their entrance into the doctrine of proportion, than any which have hitherto been alledged, and that is the difficulty of understanding and applying *Euclid's* definition of proportionable quantities. But, to take away all excuse from this quatter, I have here annexed a small dissertation, conducing (as I take it) to clear up that definition. It is an extract out of some loose papers I have by me; and therefore the reader must not be surprized if he finds some things repeated here which

276 *A Vindication of the fifth Definition* BOOK VII.  
have already been mentioned in another part of this book.

*A Vindication of the fifth Definition of the fifth Book  
of EUCLID'S Elements.*

262. *N. B.* For a more distinct understanding of what follows, it must be observed, that *By a part, in the sense of the fifth book of Euclid, is meant an aliquot part, and not a part as part related to some whole.* Thus 3 is a part of 12 in *Euclid's* sense, as being just four times contained in it; and though 9 be a part of 12 in the same sense as the part is distinguished from the whole, yet 9 in *Euclid's* sense is not a part, but parts of 12, as being three fourth parts of it.

1st. *If two quantities A and B be commensurable, then A must necessarily be either some multiple, or some part, or some parts, of B.* For if *A* and *B* be commensurable, then either *B* must measure *A*, or *A* must measure *B*, or they must both be measured by some third quantity: if *B* measures *A* any number of times, suppose 3 times, then *A* will be equal to 3 times *B*, and consequently will be a multiple of *B*: if *A* measures *B* any number of times, suppose 3 times, then *A* will be a third part of *B*, and consequently will be a part of *B*: if *A* and *B* do not measure one the other, let *C* measure them both, and let *C* be contained exactly in *A* 3 times and in *B* 4 times: then will a third part of *A* be equal to a fourth part of *B*, as being both equal to *C*; multiply both sides of the equation by 3, and you will have  $\frac{3}{3}$  of *A*, or *A* equal to  $\frac{3}{4}$  of *B*; therefore in this case *A* is said to be parts of *B*.

2dly. *If two quantities A and B are incommensurable, then A can neither be any multiple of B, nor any part or parts of it.* For if *A* was any multiple of *B*, then *B* would measure both itself and *A*, which contradicts the supposition of their incommensurability: in like manner, if *A* was any part of *B*, then *A* would measure both itself and *B*: in the last place I say that nei-  
ther

ther can  $A$  be any parts of  $B$ ; for if  $A$  was any parts of  $B$ , suppose  $\frac{1}{4}$  of  $B$ , then  $\frac{1}{4}$  of  $B$  would measure both  $A$  and  $B$ , which still contradicts the supposition:  $A$  indeed may be greater or less than some part or parts of  $B$ , but can never be equal to any; so subtil is the composition of continued quantity. As for instance; it is demonstrated in art. 201, that the side and diagonal line of a square are incommensurable to each other: let then  $A$  be the diagonal of a square whose side is  $B$ , and the square of  $A$  will be to the square of  $B$  as 2 to 1, as is evident from the 47th of the first book of *Euclid*; therefore  $A$  will be to  $B$  as the square root of 2 is to 1; but the square root of 2

is 1. 414, &c. that is,  $\frac{14}{10}$ , or more nearly  $\frac{141}{100}$ , or

more nearly still  $\frac{1414}{1000}$ : whence it follows, that if the

side of a square be divided into 10 equal parts, the diagonal will contain more than 14 of these parts, but not so much as 15 of them; if the side be divided into 100 equal parts, the diagonal will contain above 141 of such parts, but not 142; if the side be divided into 1000 equal parts, the diagonal will contain above 1414 of such parts, but not 1415; and so on *ad infinitum*: therefore the diagonal of a square can never be exactly expressed by parts of the side, any more than the side can by parts of the diagonal. The side may indeed be set off upon the diagonal, and so be considered as part of it, so far as part of the whole; but the side can never be exactly expressed by any number of aliquot parts of the diagonal, be these parts ever so small. Limits may be found and expressed by parts of the diagonal as near as possible to each other, between which the side shall always consist, and by which it may be expressed to any degree of exactness except perfect exactness\*. And thus also may approximations be made in the expressions of

\* See the Quarto Edition, p. 306.

many other incommensurable quantities one by another.

3dly, From the last section it appears, that *If two quantities A and B be incommensurable, no multiple of one can ever be equal to any multiple of the other.* For if, for instance,  $4A$  could be equal to  $3B$ , then (dividing by 4)  $A$  would be found to be just  $\frac{3}{4}$  of  $B$ , contrary to what has been above demonstrated.

4thly, *If four quantities A, B, C, and D, be such, that A is the same part or parts of B that C is of D, then are those four quantities A, B, C, and D, said to be proportionable, or A is said to have the same proportion to B that C hath to D.* Thus if  $A$  be a fourth part of  $B$ , and  $C$  a fourth part of  $D$ , then  $A$  will be the same part of  $B$  that  $C$  is of  $D$ , and they will be

proportionable. Thus again, if  $A = \frac{3}{4} B$ , and  $C =$

$\frac{3}{4} D$ , or if  $A = \frac{8}{4} B$  or  $2B$ , and  $C = \frac{8}{4} D$  or  $2D$ , or

if  $A = \frac{11}{4} B$ , and  $C = \frac{11}{4} D$ , in all these instances

(comprehending multiples under the notion of parts)  $A$  may be said to be the same parts of  $B$  that  $C$  is of  $D$ ; and therefore, according to this definition,  $A$  hath the same proportion to  $B$  that  $C$  hath to  $D$ , which is true, and the mark of proportionality here given is infallible, but not adequate to our idea of it; for though this mark be never found without proportionality, yet proportionality is often found without this mark. Proportionality is often found among incommensurables; but it can never be tried or proved by the marks here given. I believe nobody ever doubted that the side of one square hath the same proportion to its diagonal that the side of any other square hath to its diagonal; and therefore  $A$  may have the same proportion to  $B$  that  $C$  hath to  $D$ , though  $A$  be incommensurable to  $B$ , and  $C$  to  $D$ : yet who can say in this case, that  $A$  is the same part or parts of  $B$  that  $C$

is of  $D$ , when it has already been shewn, that  $A$  is no part or parts of  $B$ , nor  $C$  of  $D$ ? This way therefore of defining proportionable quantities by a similitude of aliquot parts, cannot (in strictness of Geometry) be laid down as a proper foundation, so as from thence to derive all the other properties of proportionality: for since these properties are to be applied to incommensurable as well as commensurable quantities, it is fit they should be deduced from a fundamental property that relates equally to both.

5thly, In order then to establish a more general character of proportionality, I shall assume the following principle, which equally relates to commensurable and incommensurable quantities; and which, I believe, there is no one who has a just idea of proportionality, which way soever he may choose to express it, or whether he can express it or not, but will easily allow me, which is, that *If four quantities  $A$ ,  $B$ ,  $C$ , and  $D$ , be proportionable, that is, if  $A$  has the same proportion to  $B$  that  $C$  hath to  $D$ , it will then be impossible for  $A$  to be greater than any part or parts of  $B$ , but  $C$  must also be greater than a like part or parts of  $D$ ; or for  $A$  to be equal to any part or parts of  $B$ , but that  $C$  must also be equal to a like part or parts of  $D$ ; or for  $A$  to be less than any part or parts of  $B$ , but that  $C$  must also be less than a like part or parts of  $D$ .* Thus if  $A$  hath the same proportion to  $B$  that  $C$  hath to  $D$ , it will then be impossible for  $A$  to be greater than, equal to, or less than,  $\frac{14}{10}$  of  $B$ , but  $C$  must also

be greater than, equal to, or less than,  $\frac{14}{10}$  of  $D$ . This principle, I say, is so very clear that nothing more needs to be said of it, either by way of explication or demonstration: and if by the help hereof I can demonstrate the converse, we shall then have a general mark of proportionality as extensive as proportionality itself. Now the converse of the foregoing proportion is this; *If there*

be four quantities  $A, B, C$ , and  $D$ , and if the nature of these quantities be such, that  $A$  cannot possibly be greater than, equal to, or less than, any part or parts of  $B$ , but at the same time  $C$  must necessarily be greater than, equal to, or less than, a like part or parts of  $D$ , let the number or denomination of these parts be what they will; I say then, that  $A$  must necessarily have the same proportion to  $B$  that  $C$  hath to  $D$ . If this be denied, let some other quantity  $E$  have the same proportion to  $D$  that  $A$  hath to  $B$ , that is, let  $A, B, E$ , and  $D$ , be proportionable quantities; then imagining the quantity  $D$  to be divided into any number of equal parts, suppose 10, let  $E$  be greater than 14 of these parts and less than 15, that is, let  $E$  be greater than  $\frac{14}{10}$  and

less than  $\frac{15}{10}$  of  $D$ ; then must  $A$  necessarily be greater

than  $\frac{14}{10}$  and less than  $\frac{15}{10}$  of  $B$ : this is evident from the concession already made, since  $A$  is supposed to have the same proportion to  $B$  that  $E$  hath to  $D$ . But

if  $A$  be greater than  $\frac{14}{10}$  and less than  $\frac{15}{10}$  of  $B$ , then

$C$  must be greater than  $\frac{14}{10}$  and less than  $\frac{15}{10}$  of  $D$  by the hypothesis; the relation between  $A, B, C$ , and  $D$ , being supposed to be such, that  $A$  cannot be greater or less than any part or parts of  $B$ , but  $C$  accordingly must be greater or less than a like part or parts of  $D$ . Therefore we are now advanced thus far, that if  $E$

lies between  $\frac{14}{10}$  and  $\frac{15}{10}$  of  $D$ ,  $C$  must also necessarily lie betwixt the same limits; now the difference betwixt

$\frac{14}{10}$  and  $\frac{15}{10}$  of  $D$  is  $\frac{1}{10}$  of  $D$ ; therefore the difference

betwixt  $C$  and  $E$ , which lie both between these two limits,

limits, must be less than  $\frac{1}{10}$  of  $D$ . This is upon a supposition that the quantity  $D$  was at first divided into 10 equal parts; but if instead of 10 we had supposed it to have been divided into 100, or 1000, or 10000, equal parts (which suppositions could not have affected the quantities  $C$  and  $E$ ), the conclusion would then have been, that the difference betwixt  $C$  and  $E$  would have been less than the hundredth, or thousandth, or ten thousandth, part of  $D$ ; and so on *ad infinitum*: therefore the difference between  $C$  and  $E$  (if there be any difference) must be less than any part of  $D$  whatever; therefore the difference between  $C$  and  $E$  is only imaginary, and not real; therefore in reality  $C$  is equal to  $E$ . Since then  $C$  is equal to  $E$ , and that  $A$  is to  $B$  as  $E$  is to  $D$ , the consequence must be that  $A$  is to  $B$  as  $C$  is to  $D$ . Q. E. D.

Here then we have a proper characteristic of proportionality which always accompanies it, and, on the other hand, it is never to be found without it, to wit, that four quantities may be said to be proportionable, the first to the second as the third is to the fourth, when the first cannot be greater than, equal to, or less than, any part or parts of the second, but the third must accordingly be greater than, equal to, or less than, a like part or parts of the fourth: or thus; *Four quantities may be said to be proportionable as above, when the first cannot be contained between two limits expressed by any parts of the second, how near soever these limits may approach to each other, but the third must necessarily be contained between the limits expressed by like parts of the fourth.*

6thly, Had *Euclid* stopped here, without refining any further upon the criterion of proportionality delivered in the last section (for I dare venture to affirm, he was no stranger to it), I doubt not but it would have given much greater satisfaction to the generality of his disciples, especially those of a less delicate taste, than

than that which he advances in the fifth book of his Elements, as being more closely connected with the common idea of proportionality: but it was easy to see, that in demonstrating several other affections of proportionable quantities upon this scheme, there would then be frequent occasion for taking such and such parts of magnitudes, as there is now for taking such and such multiples of them, the *praxis* of which partition had no where as yet been taught by *Euclid*; nay, he rather seems to have determined, as far as possible, to avoid it, and that upon no ill grounds neither; for the use of whole numbers is in all cases justly esteemed more natural and more elegant than that of fractions, and the multiplication of quantities has always been looked upon as more simple in the conception than the resolution of them into their aliquot parts. It is for this reason that *Euclid* never shews how to multiply a line or any other quantity whatever, assuming the *praxis* thereof as a sort of *postulatum*; whereas in the ninth proposition of the sixth book of his Elements he shews how to cut off any aliquot part of any given line whatever. Upon these and such like considerations it was that *Euclid* resolved to advance his characteristic property of proportionality one step higher, by substituting multiples instead of aliquot parts in such a manner as we shall now describe; and we shall at the same time demonstrate the justness of his definition from what has been already laid down in the last section. The proposition to be demonstrated shall be this: *If there be four quantities, A, B, C, and D, whereof EA and EC are any equimultiples of the first and third, and FB and FD are any other equimultiples of the second and fourth; and if now these quantities are of such a nature, that EA cannot be greater than, equal to, or less than, FB, but at the same time EC must necessarily be greater than, equal to, or less than, FD, when compared respectively, be the multipliers E and F what they will: I say then that A must necessarily have the same proportion to B that C*

*hath*

*path to D.* Now that four quantities may be under such circumstances as are here described, can be questioned by no one who has with any attention considered the nature of proportionable quantities: for suppose  $A$  to be the diameter and  $B$  the circumference of any circle, and  $C$  to be the diameter and  $D$  the circumference of any other circle; who doubts but that twenty-two times the diameter of one circle will be greater than, equal to, or less than, seven times the circumference, according as twenty-two times the diameter of the other circle is greater than, equal to, or less than, seven times the circumference of that circle? I now proceed to the demonstration of the proposition.

If it be denied that  $A$  is to  $B$  as  $C$  is to  $D$ , let  $A$  be to  $B$  as  $G$  is to  $D$ ; and then, supposing  $D$  to be divided into 10 equal parts, let  $G$  be greater than 14 of these parts, and less than 15: then since by the supposition  $A$  is to  $B$  as  $G$  is to  $D$ , we shall have  $A$  greater than  $\frac{14}{10}$  and less than  $\frac{15}{10}$  of  $B$ ; therefore  $10A$  will be greater than  $14B$  and less than  $15B$ ; but by the *hypothesis*, no multiple of  $A$  can be greater or less than any multiple of  $B$ , but the same multiple of  $C$  must be greater or less than the same multiple of  $D$ ; therefore  $10C$  is greater than  $14D$  and less than  $15D$ ; therefore  $C$  is greater than  $\frac{14}{10}$  and less than  $\frac{15}{10}$  of  $D$ ;

therefore if  $G$  be a quantity between  $\frac{14}{10}$  and  $\frac{15}{10}$  of  $D$ ,  $C$  must also be a quantity between the same limits; therefore the difference betwixt  $C$  and  $G$  must be less than  $\frac{1}{10}$  of  $D$ . This is upon a supposition that  $D$  was divided into 10 equal parts; but  $C$  and  $G$  will be the same, into what number of parts soever we suppose  $D$  to be divided; therefore if we suppose  $D$  to be divided into 100,

100, 1000, or 10000, equal parts, &c. the difference betwixt *C* and *G* might have been shewn to be less than the hundredth, or the thousandth, or the ten thousandth, part of *D*; and so on *ad infinitum*; therefore *C* and *G* are equal, as was shewn in the 5th section. Since then *A* cannot be greater than, equal to, or less than, any part or parts of *B*, but *G* must be greater than, equal to, or less than, a like part or parts of *D*, because *A* is to *B* as *G* is to *D*; and since *G* cannot be greater than, equal to, or less than, any part or parts of *D*, but *C* must be greater than, equal to, or less than, the same part or parts of *D*, because *G* and *C* are equal; it follows *ex æquo*, that *A* cannot be greater than, equal to, or less than, any part or parts of *B*, but that *C* must accordingly be greater than, equal to, or less than, a like part or parts of *D*; and consequently that *A* is to *B* as *C* is to *D*, according to the mark of proportionality given in the last section. *Q. E. D.*

*Four quantities then may be said to be proportionable, the first to the second as the third to the fourth, when no equimultiples whatever can be taken of the first and third, but what must either be both greater than; or both equal to, or both less than, any other equimultiples that can possibly be taken of the second and fourth, when compared respectively.*

7thly, As a number is a discrete, and not a continued quantity, there is such a thing as a *minimum* in the parts of number, whereas in those of extension there is none; whence it follows, that the parts of number must necessarily be more distinct, and for that reason more assignable, than are the parts of extension. Again, as all numbers are commensurable by unity, every number may be conceived either as some multiple, or some part, or some parts, of every other. Hence it is that *Euclid*, defining proportionable numbers, makes use of the definition given in the 4th section; so unwilling was he to recede from

the common notion of proportionable quantities, whenever the subject he treated of would bear it.

*Of the seventh Definition of the fifth Book of EUCLID.*

263. If it be allowed to be a sufficient mark of the proportionality of four quantities, when they are so related to one another in their own natures, that no equimultiples can be taken of the first and third, but what must either be both greater than, or both equal to, or both less than, any other equimultiples that can possibly be taken of the second and fourth; then wherever it happens, or may happen otherwise, there can be no proportionality. As for instance, *If in comparing equimultiples of the first and third with other equimultiples of the second and fourth, there be any cases wherein the first multiple shall be greater than the second, and yet the third not greater than the fourth; or wherein the first multiple shall be less than the second, and the third not less than the fourth; then the first quantity will not have the same proportion to the second that the third hath to the fourth, but either a greater as in the former case, or a less as in the latter.* Nay, and I may add further, that if of four quantities, the first hath a greater proportion to the second than the third hath to the fourth, there must be cases existing, whether those cases can be assigned or not, wherein of equimultiples of the first and third, and of other equimultiples of the second and fourth, the first multiple shall exceed the second, and yet the third shall not exceed the fourth: for if no such cases were possible, then the first quantity must either have the same proportion to the second that the third hath to the fourth, or a less: both which are contrary to the supposition. Thus we have found the fifth and seventh definitions of the fifth book of the Elements both of a piece.

*A Question arising out of the foregoing Article.*

264. This is all that was necessary to be observed concerning the foregoing definitions; but if, having given four quantities  $A$ ,  $B$ ,  $C$ , and  $D$ , whereof  $A$  hath a greater proportion to  $B$  than  $C$  hath to  $D$ , any one, for his own private satisfaction, would know how to find such equimultiples of  $A$  and  $C$ , and such other equimultiples of  $B$  and  $D$ , that  $A$ 's multiple shall exceed that of  $B$ , and at the same time  $C$ 's multiple shall not exceed that of  $D$ , it must be done thus: If the quantities  $A$ ,  $B$ ,  $C$ , and  $D$ , be commensurable, let their ratios be expressed by numbers: as for instance, let  $A$  be to  $B$  as 7 to 5, and let  $C$  be to  $D$  as 4 to 3; then will 4 and 3, the numeral expressions of the lesser ratio, be the multiplicators required, if of the terms  $A$  and  $B$ , the greater term  $A$  be multiplied into the lesser multiplicator 3, and the lesser term  $B$  into the greater multiplicator 4; for then  $3A$  (21) will be greater than  $4B$  (20), and yet  $3C$  (12) will not be greater than  $4D$  (12), for the two last multiples are equal. But if such multiples be required, that the first multiple shall be greater than the second, and at the same time the third multiple shall be less than the fourth, then some intermediate fraction must be taken between  $\frac{7}{5}$  and  $\frac{4}{3}$ , and the terms of such a fraction will be the multiplicators required. As for instance, throwing the extreme fractions into decimals, we have  $\frac{7}{5} = 1.4$ , and  $\frac{4}{3} = 1.34 -$ ; therefore if any decimal fraction be taken between 1.4 and 1.34, such a fraction being reduced to integral terms will give the multiplicators required. Let us assume 1.375, that is  $\frac{1375}{1000}$  or  $\frac{11}{8}$ ; then will  $8A$  (56) be greater than  $11B$  (55), and at the same time  $8C$  (32) will be less than  $11D$  (33).

If the quantity  $A$  be incommensurable to  $B$ , or  $C$  to  $D$ , or both to both, find however, by scholium the

the second in art. 179 \*, such numbers as will express these ratios as accurately as occasion requires. As let the ratio of the number  $E$  to the number  $F$  be nearly the same with that of  $A$  to  $B$ , and let the ratio of the number  $G$  to the number  $H$  be nearly the same with that of  $C$  to  $D$ ; then if either of these ratios, to wit, the ratio of  $E$  to  $F$ , or the ratio of  $G$  to  $H$ , lie between the ratios of  $A$  to  $B$  and of  $C$  to  $D$ , the terms of the intermediate ratio will make proper multipliers, but if neither of these cases happen, some intermediate fraction must be taken between the two frac-

tions  $\frac{E}{F}$  and  $\frac{G}{H}$ .

Having thus prepared my young student for *Euclid's* doctrine of proportion, partly by setting him right in his notions of things, and partly by removing out of his way all that rubbish which seemed to block up his entrance to it; I hope I shall now be able to conduct him through the whole with a great deal of ease, and then he will meet with fewer difficulties in reading the following propositions than an equal number in any other part of the Elements: and yet all I have done herein has been only to mitigate, as far as I thought proper, the rigour and severity of the author's manner of writing, and to render his demonstrations more easy to the imagination, which the compiler in his whole system seems to have had no great tenderness for: but, whatever I have done else, I have taken care to preserve the force of the demonstrations, and I hope, in a great measure, their elegance too. I have used no algebraic computations in demonstrating these propositions, except what may be justified by the antecedent ones; as well knowing that these principles were never intended to depend upon arithmetical operations, but rather arithmetical operations upon them. I have however, for the reader's ease, made use of the simplest algebraic notation. Thus  $A, B, C, D$ , signify magnitudes of any kind whatever;  $E, F, G, H$ ,

\* See the Quarto Edition, p. 285.

always signify whole numbers, unless where notice is given to the contrary:  $A + B$  signifies the sum of any two homogeneous magnitudes  $A$  and  $B$ ;  $A - B$  their difference, or the excess of  $A$  above  $B$ ;  $EA$  and  $FB$  signify any two multiples of  $A$  and  $B$ , the multipliers being  $E$  and  $F$ ; &c. I have sometimes also used very easy consequences of this notation; as that if  $A - B$  be added to  $B$ , the sum will be  $A$ , which indeed is a general axiom, and saying no more than that if to any magnitude be added the excess of a greater above it, the sum will be the greater magnitude.

## The Fifth Book of EUCLID'S ELEMENTS.

### DEFINITIONS.

265. 1. *A lesser magnitude is said to be a part of a greater, when the lesser measures the greater.*

2. *A greater magnitude is said to be a multiple of a less, when the greater is measured by the less.*

*Note.* Our language is not nice enough to express these two definitions as they are in the Greek and Latin.

We may further observe, that by these two definitions every simple quantity is excluded from being considered either as a part or multiple of itself; for to be a part, in this sense, is to be less than that whereof it is a part, and to be a multiple is to be greater than that whereof it is a multiple.

3. *Ratio is that mutual relation two homogeneous quantities are in, when compared together in respect to their quantity.* Thus the excess of 2 above 1 is equal to the excess of 4 above 3, and yet the ratio of 2 to 1 is greater than the ratio of 4 to 3; that is, 2 has more magnitude when compared with 1 than 4 hath when compared with 3; since 2 is double of 1, and 4 is not double of 3. But on the other hand, 3 hath a greater ratio to 4 than 1 hath to 2, because 3 hath more magnitude in comparison of 4 than 1 hath in comparison of 2.

comparison of 2; for 3 is more than the half of 4, whereas 1 is but just the half of 2.

4. *All quantities are said to be in some ratio or other, when they are capable of being so multiplied as to exceed one another.*

*Note.* By this definition, 1st, All heterogeneous quantities are excluded from having any ratio one to another, because heterogeneous quantities are such, that their multiples are no more capable of comparison as to excess and defect, than the quantities themselves: a yard can never be multiplied till it exceeds an hour, &c. 2dly, All infinitely small quantities are hereby excluded from having any ratio to finite ones, because the former can never be so multiplied as to exceed the latter.

5. *Magnitudes are said to be in the same ratio, the first to the second as the third to the fourth, when no equimultiples can be taken of the first and third, but what must either be both greater than, or both equal to, or both less than, any other equimultiples that can possibly be taken of the second and fourth.*

*Note.* This and the seventh definition have been explained already.

6. *Magnitudes in the same ratio may be called proportionals.*

7. *If there be four quantities, whereof equimultiples are taken of the first and third, and other equimultiples of the second and fourth; and if any case can be assigned, wherein the multiple of the first shall be greater than the multiple of the second, and at the same time the multiple of the third shall not be greater than the multiple of the fourth; then of these four quantities, the first is said to have a greater ratio to the second than the third hath to the fourth.*

8. *Proportion consists in a similitude of ratios.*

9. *Proportion cannot be expressed in fewer than three terms: as when we say that A is to B as B is to C.*

10. *Whenever three quantities are continual proportionals, the first is said to be to the third in a duplicate*

ratio of the first to the second: and on the other hand, the first is said to be to the second in a subduplicate ratio of the first to the third.

11. If four quantities be continual proportionals, the first is said to be to the fourth in a triplicate ratio of the first to the second; and so on.

12. The antecedents of all proportions are called homologous terms; and so also are the consequents: but antecedents and consequents considered together are never called homologous terms, but heterologous.

Note. These three last definitions, though placed here, have nothing to do in the following fifth book, but in the sixth.

13. Alternate proportion is, when four quantities being proportionable, the first to the second as the third to the fourth, it is concluded, that the first is to the third as the second to the fourth; the justness of which conclusion, as well as of all the rest that follow, will be sufficiently made out in the following propositions.

14. Inverse proportion is, when four quantities being proportionable, the first to the second as the third to the fourth, it is concluded, that the second is to the first as the fourth to the third.

15. Composition of proportion is, when four quantities being proportionable, the first to the second as the third to the fourth, it is concluded, that the sum of the first and second is to the second as the sum of the third and fourth is to the fourth,

16. Division of proportion is, when four quantities being proportionable, the first to the second as the third to the fourth, it is concluded, that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.

17. Conversion of proportion is, when four quantities being proportionable, the first to the second as the third to the fourth, it is concluded, that the first is to the excess of the first above the second as the third is to the excess of the third above the fourth.

18. If

18. If ever so many quantities in one series be compared with as many in another; and if from all the ratios in one being equal to all those in the other, either in the same or a different order, it be concluded, that the extremes in one series are in the same proportion with the extremes in the other, this proportionality of the extremes is said to follow *ex æquo*, or *ex æqualitate rationum*.

19. If all the ratios in one series be equal to all those in the other, and in the same order, this is called *ordinate proportion*; and the extremes in this case are said to be proportionable *ex æquo ordinate*, or barely *ex æquo*.

20. If all the ratios in one series be equal to all those in the other, but not in the same order, this is called *inordinate proportion*; and the extremes are said to be proportionable *ex æquo perturbate*.

Thus if *A*, *B*, and *C*, in one series be compared with *D*, *E*, and *F*, in another; and if *A* is to *B* as *D* to *E*, and *B* to *C* as *E* to *F*, this is called *ordinate proportion*, and *A* is said to be to *C* as *D* to *F* *ex æquo ordinate*, or barely *ex æquo*: but if *A* is to *B* as *E* to *F*, and *B* to *C* as *D* to *E*, this is called *inordinate proportion*, and *A* is said to be to *C* as *D* to *F* *ex æquo perturbate*.

# PROPOSITION I.

266. If there be ever so many homogeneous quantities, *A*, *B*, *C*, whereof *EA*, *EB*, *EC*, are equimultiples respectively; I say then, that the sum  $\overline{EA + EB + EC}$  will be the same multiple of the sum  $\overline{A + B + C}$  that *EA* is of *A*, or *EB* of *B*, &c.

For the multiples *EA*, *EB*, and *EC*, may be considered as so many distinct heaps or parcels, whereof *EA* consists wholly of *A*'s, *EB* of *B*'s, and *EC* of *C*'s; and since the numbers of *A*'s in *EA* is the same with the number of *B*'s in *EB*, or of *C*'s in *EC*, it follows, that as often as *A* can be singly taken out of *EA*, or

$B$  out of  $EB$ , or  $C$  out of  $EC$ , just so often may the whole sum  $\overline{A+B+C}$  be taken out of the whole sum  $\overline{EA+EB+EC}$ ; therefore the sum  $\overline{EA+EB+EC}$  is the same multiple of the sum  $\overline{A+B+C}$  that  $EA$  is of  $A$ , or  $EB$  of  $B$ , &c.  $\mathcal{Q}. E. D.$

## PROPOSITION 2.

267. *If  $EA$  and  $EB$  be equimultiples of any two quantities whatever  $A$  and  $B$ , and if  $FA$  and  $FB$  be also equimultiples of the same; I say then that the sum  $\overline{EA+FA}$  will be the same multiple of  $A$  that the sum  $\overline{EB+FB}$  is of  $B$ .*

For since the number of  $A$ 's in  $EA$  is the same with the number of  $B$ 's in  $EB$ ; and since also the number of  $A$ 's in  $FA$  is the same with the number of  $B$ 's in  $FB$ , add equals to equals, and the number of  $A$ 's in  $\overline{EA+FA}$  will be the same with the number of  $B$ 's in  $\overline{EB+FB}$ , that is, the sum  $\overline{EA+FA}$  will be the same multiple of  $A$  that the sum  $\overline{EB+FB}$  is of  $B$ .  $\mathcal{Q}. E. D.$

## PROPOSITION 3.

268. *If  $EA$  and  $EB$  be equimultiples of any two quantities whatever  $A$  and  $B$ , and if  $3EA$  and  $3EB$  be any equimultiples of  $EA$  and  $EB$ ; I say then, that  $3EA$  and  $3EB$  will also be equimultiples of  $A$  and  $B$ .*

This is evident from the last proposition: for since  $EA$  and  $EB$  are equimultiples of  $A$  and  $B$ ; and since  $EA$  and  $EB$  are again equimultiples of the same, it follows from that proposition, that the sum  $2EA$  is the same multiple of  $A$  that the sum  $2EB$  is of  $B$ : again, since  $2EA$  and  $2EB$  are equimultiples of  $A$  and  $B$ , and since  $EA$  and  $EB$  are other equimultiples of the same, the sum  $3EA$  is the same multiple of  $A$  that the sum  $3EB$  is of  $B$ ; and so on *ad infinitum*.  $\mathcal{Q}. E. D.$

PROPOSITION 4.

269. *If four quantities A, B, C, and D, be proportionable, A to B as C to D, and if EA and EC be any equimultiples of the first and third, and FB and FD any other equimultiples of the second and fourth; I say then that these multiples will also be proportionable, provided they be taken in the same order as the proportionable quantities whereof they are multiples; that is, that EA will be to FB as EC is to FD.*

For let  $3EA$  and  $3EC$  be any equimultiples of  $EA$  and  $EC$ , and let  $2FB$  and  $2FD$  be any other equimultiples of  $FB$  and  $FD$ : then since  $3EA$  and  $3EC$  are equimultiples of  $EA$  and  $EC$ , and since  $EA$  and  $EC$  are equimultiples of  $A$  and  $C$ , it follows from the last proposition that  $3EA$  and  $3EC$  are equimultiples of  $A$  and  $C$ ; and for the same reason  $2FB$  and  $2FD$  are also equimultiples of  $B$  and  $D$ . Since then, *ex hypothesi*,  $A$  is to  $B$  as  $C$  is to  $D$ ; and since  $3EA$  and  $3EC$  are equimultiples of  $A$  and  $C$ , and  $2FB$  and  $2FD$  are also other equimultiples of  $B$  and  $D$ , it follows from the fifth definition, that  $3EA$  cannot be greater than, equal to, or less than,  $2FB$ , but  $3EC$  must also be greater than, equal to, or less than,  $2FD$ . Again, since we have four quantities  $EA, FB, EC, FD$ , whereof  $3EA$  and  $3EC$  represent any equimultiples of the first and third, and  $2FB$  and  $2FD$  any other equimultiples of the second and fourth; and since  $3EA$  cannot be greater than, equal to, or less than,  $2FB$ , but  $3EC$  must in like manner be greater than, equal to, or less than,  $2FD$ , it follows from the fifth definition, that these four quantities  $EA, FB, EC, FD$ , are proportionable; that  $EA$  is to  $FB$  as  $EC$  to  $FD$ . Q. E. D.

SCHOLIUM.

To this place is usually referred the inversion of proportion (though why to this, rather than to any other, I know not); that is, that *if four quantities*

be proportionable, they will also be inversely proportionable: as if  $A$  be to  $B$  as  $C$  is to  $D$ , then  $B$  will be to  $A$  as  $D$  to  $C$ . For let  $EA$  and  $EC$  be any equimultiples of  $A$  and  $C$ , and let  $FB$  and  $FD$  be any other equimultiples of  $B$  and  $D$ ; and first let us suppose  $FB$  to be greater than  $EA$ ; then will  $EA$  be less than  $FB$ : and because  $A$  is to  $B$  as  $C$  is to  $D$ ,  $EC$  will also be less than  $FD$  by the fifth definition; and therefore  $FD$  will be greater than  $EC$ : thus then we see that if  $FB$  be greater than  $EA$ ,  $FD$  will also be greater than  $EC$ . And after the same manner it may be demonstrated, that if  $FB$  be equal to, or less than,  $EA$ ,  $FD$  in like manner will be equal to, or less than,  $EC$ . Since then we have four quantities  $B, A, D, C$ , whereof  $FB$  and  $FD$  are equimultiples of the first and third, and  $EA$  and  $EC$  are other equimultiples of the second and fourth; and since  $FB$  cannot be greater than, equal to, or less than,  $EA$ , but  $FD$  must accordingly be greater than, equal to, or less than,  $EC$ , it follows from the fifth definition, that these four quantities  $B, A, D, C$ , must be proportionable; that  $B$  must be to  $A$  as  $D$  to  $C$ . Q. E. D.

### PROPOSITION 5.

270. If  $A$  and  $B$  be any two homogeneous quantities, whereof  $A$  is the greater, and whereof  $EA$  and  $EB$  are equimultiples respectively; I say then that the difference  $EA - EB$  will be the same multiple of the difference  $A - B$  that  $EA$  is of  $A$ , or  $EB$  of  $B$ .

If this be denied, let  $G$  be the same multiple of  $A - B$  that  $EA$  is of  $A$ , or  $EB$  of  $B$ ; then we shall have two quantities  $A - B$  and  $B$ , whose sum is  $A$ , and whereof  $G$  and  $EB$  are equimultiples respectively; therefore, by the first proposition, the sum  $G + EB$  will be the same multiple of the sum  $A$  that  $EB$  is of  $B$ : but  $EA$  is also the same multiple of  $A$  that  $EB$  is of  $B$ ; therefore  $G + EB$  is the same multiple

multiple of  $A$  that  $EA$  is of  $A$ ; therefore  $\overline{G+EB}$  must be equal to  $EA$ ; take  $EB$  from both sides, and  $G$  will be equal to  $\overline{EA-EB}$ ; but  $G$  was the same multiple of  $\overline{A-B}$  that  $EA$  was of  $A$ , or  $EB$  of  $B$ ; therefore  $\overline{EA-EB}$  will be the same multiple of  $\overline{A-B}$  that  $EA$  is of  $A$ , or  $EB$  of  $B$ . Q. E. D.

# PROPOSITION 6.

271. *If from  $EA$  and  $EB$ , equimultiples of any two quantities  $A$  and  $B$ , be subtracted  $FA$  and  $FB$ , any other equimultiples of the same; the remainders  $\overline{EA-FA}$  and  $\overline{EB-FB}$  will either be equal to the quantities  $A$  and  $B$  respectively, or they will be equimultiples of them.*

## CASE I.

In the first place, let the remainder  $\overline{EA-FA}$  be equal to  $A$ ; I say then that the other remainder  $\overline{EB-FB}$  will also be equal to  $B$ . For since  $FA$  is the same multiple of  $A$  that  $FB$  is of  $B$ , it follows from the nature of multiples, that  $\overline{FA+A}$  will be the same multiple of  $A$  that  $\overline{FB+B}$  is of  $B$ : but  $A$  is equal to  $\overline{EA-FA}$ ; and adding  $FA$  to both sides we have  $\overline{FA+A=EA}$ ; therefore instead of saying as before, that  $\overline{FA+A}$  is the same multiple of  $A$  that  $\overline{FB+B}$  is of  $B$ , we may now say that  $EA$  is the same multiple of  $A$  that  $\overline{FB+B}$  is of  $B$ : but  $EA$  is the same multiple of  $A$  that  $EB$  is of  $B$ ; therefore  $EB$  is the same multiple of  $B$  that  $\overline{FB+B}$  is of  $B$ ; therefore  $EB$  is equal to  $\overline{FB+B}$ ; subtract  $FB$  from both sides, and you will have  $\overline{EB-FB=B}$ . Q. E. D.

## C A S E 2.

Let us now suppose the remainder  $\overline{EA - FA}$  to be some multiple of  $A$ ; for if  $A$  measures both  $EA$  and  $FA$ , it must measure  $\overline{EA - FA}$ ; and so  $\overline{EA - FA}$  must be some multiple of  $A$ ; and for the same reason, the other remainder  $\overline{EB - FB}$  must be some multiple of  $B$ : I say then in the next place, that  $\overline{EB - FB}$  must be the same multiple of  $B$  that  $\overline{EA - FA}$  is of  $A$ . If this be denied, let  $G$  be the same multiple of  $B$  that  $\overline{EA - FA}$  is of  $A$ ; then since  $\overline{EA - FA}$  and  $G$  are equimultiples of  $A$  and  $B$ , and since  $FA$  and  $FB$  are also other equimultiples of the same, it follows from the second proposition, that the sum  $\overline{EA - FA + FA}$  will be the same multiple of  $A$  that  $\overline{G + FB}$  is of  $B$ : but  $\overline{EA - FA + FA} = EA$ ; therefore  $EA$  is the same multiple of  $A$  that  $\overline{G + FB}$  is of  $B$ : but  $EA$  is the same multiple of  $A$  that  $EB$  is of  $B$ ; therefore  $EB$  is the same multiple of  $B$  that  $\overline{G + FB}$  is of  $B$ ; therefore  $EB$  is equal to  $\overline{G + FB}$ ; therefore  $\overline{EB - FB}$  is equal to  $G$ : but  $G$  was the same multiple of  $B$  that  $\overline{EA - FA}$  is of  $A$  by the supposition; therefore  $\overline{EB - FB}$  is the same multiple of  $B$  that  $\overline{EA - FA}$  is of  $A$ . Q. E. D.

## S C H O L I U M.

As in the second definition it was provided that no simple quantity be considered as a multiple of itself, so in this proposition care is taken that no two simple quantities be considered as equimultiples of themselves; which indeed is but a consequence of that definition, and is the reason why this proposition resolves itself into two cases.

For

For a better understanding and remembering the structure of the six foregoing propositions, it may be observed, that the two last propositions are nothing else but the two first with their signs changed. In the first proposition it was demonstrated, that the sum  $\overline{EA+EB}$  is the same multiple of the sum  $\overline{A+B}$  that  $\overline{EA}$  is of  $A$ , or  $\overline{EB}$  of  $B$ : in the fifth proposition it is demonstrated, that the difference  $\overline{EA-EB}$  is the same multiple of the difference  $\overline{A-B}$  that  $\overline{EA}$  is of  $A$  or  $\overline{EB}$  of  $B$ . Again, in the second proposition it was demonstrated, that the sum  $\overline{EA+FA}$  is the same multiple of  $A$  that the sum  $\overline{EB+FB}$  is of  $B$ ; and in the sixth it is demonstrated that the remainder  $\overline{EA-FA}$  is the same multiple of  $A$  that the remainder  $\overline{EB-FB}$  is of  $B$ .

# PROPOSITION 7.

272. *If two equal quantities A and B be compared with a third as C, I say then, that both A and B will have the same proportion to C; and vice versa, that C will have the same proportion both to A and to B.*

For taking any equimultiples of  $A$  and  $B$ , suppose  $3A$  and  $3B$ , and any other multiple of  $C$ , suppose  $5C$ , it is plain that  $3A$  must be equal to  $3B$ , because  $A$  is equal to  $B$ : but if  $3A$  be equal to  $3B$ , then it will be impossible for  $3A$  to be greater than, equal to, or less than,  $5C$ , but  $3B$  must accordingly be greater than, equal to, or less than, the same  $5C$ ; therefore we have four quantities  $A$ ,  $C$ ,  $B$ , and  $C$ , whereof  $3A$  and  $3B$  represent any equimultiples of the first and third, and  $5C$  and  $5C$  any other equimultiples of the second and fourth; and since the first multiple  $3A$  cannot be greater than, equal to, or less than, the second  $5C$ , but the third multiple  $3B$  must accordingly be greater than, equal to, or less than, the fourth  $5C$ , it follows from the fifth definition, that these four quantities

quantities  $A$ ,  $C$ ,  $B$ , and  $C$ , are proportionable,  $A$  to  $C$  as  $B$  to  $C$ . *Q. E. D.*

Again, since  $3A$  is equal to  $3B$ , it will be impossible for  $5C$  to be greater than, equal to, or less than,  $3A$ , but the same  $5C$  must also be greater than, equal to, or less than,  $3B$ ; therefore we have four quantities  $C$ ,  $A$ ,  $C$ , and  $B$ , whereof  $5C$  and  $5C$  represent any equimultiples of the first and third, and  $3A$  and  $3B$  any other equimultiples of the second and fourth; and since the first multiple  $5C$  cannot be greater than, equal to, or less than, the second  $3A$ , but the third multiple  $5C$  must also be greater than, equal to, or less than, the fourth  $3B$ , it follows from the fifth definition, that these four quantities  $C$ ,  $A$ ,  $C$ , and  $B$ , must be proportionable,  $C$  to  $A$  as  $C$  to  $B$ . *Q. E. D.*

### PROPOSITION 8.

273. *If two unequal quantities A and B, whereof A is the greater, be compared with a third as C, I say then that A will have a greater proportion to C than B hath to C; but that, on the other hand, C will have a greater proportion to B than it hath to A.*

For since, by the supposition,  $A$  is greater than  $B$ ,  $A - B$  will be the excess of  $A$  above  $B$ ; and by the fifth proposition, if  $EB$  be any multiple of  $B$ ,  $EA - EB$  will be the same multiple of  $A - B$ : multiply then these two quantities  $B$  and  $A - B$  alike, till of the equimultiples thence arising, the less shall be greater than  $C$ ; then will the other be much greater; let these equimultiples be  $3B$  and  $3A - 3B$ , each being greater than  $C$ : lastly multiply  $C$  till you come to a multiple of it that shall be the next greater than  $3B$ , which multiple let be  $5C$ ; then it is plain that  $3B$  cannot be less than  $4C$ ; for if it was, then  $4C$ , and not  $5C$ , would be the next multiple of  $C$  greater than  $3B$ , contrary to the supposition. Since then  $3B$  cannot be less than  $4C$ ; it follows, that if to  $3B$  be added a greater

greater quantity, and to  $4C$  a less, the former sum will be greater than the latter : but  $3A - 3B$  is greater than  $C$  by the construction ; add then  $3A - 3B$  to  $3B$ , and  $C$  to  $4C$ , and you will have  $3A$  greater than  $5C$  : but  $3B$  is less than  $5C$  by the construction ; therefore we have four quantities  $A$ ,  $C$ ,  $B$  and  $C$ , whereof  $3A$  and  $3B$  are equimultiples of the first and third, and  $5C$  and  $5C$  are other equimultiples of the second and fourth ; and since the first multiple  $3A$  is greater than the second  $5C$ , and at the same time the third multiple  $3B$  is not greater than the fourth  $5C$ , but less, it follows from the seventh definition, that of the four quantities  $A$ ,  $C$ ,  $B$ , and  $C$ ,  $A$  hath a greater proportion to  $C$  than  $B$  hath to  $C$ . Q. E. D.

Again, since we have four quantities  $C$ ,  $B$ ,  $C$ , and  $A$ , whereof  $5C$  and  $5C$  are equimultiples of the first and third, and  $3B$  and  $3A$  are other equimultiples of the second and fourth ; and since the first multiple  $5C$  is greater than the second  $3B$ , and at the same time the third multiple  $5C$  is not greater than the fourth  $3A$ , but less, it follows from the seventh definition, that of the four quantities  $C$ ,  $B$ ,  $C$ , and  $A$ ,  $C$  hath a greater proportion to  $B$  than  $C$  hath to  $A$ . Q. E. D.

### PROPOSITION 9.

274. *If two quantities A and B have both the same proportion to a third as C, or if C hath the same proportion to both A and B ; in either of these cases A and B must be equal to each other.*

For should either of them be greater than the other, should  $A$  be greater than  $B$ , then by the last proposition,  $A$  must have a greater proportion to  $C$  than  $B$  hath to  $C$ , contrary to the first supposition ; and  $C$  must have a greater proportion to  $B$  than it hath to  $A$ , contrary to the second supposition ; therefore  $A$  and  $B$  must be equal to each other. Q. E. D.

## PROPOSITION 10.

275. *If of three quantities A, B, and C, A hath a greater proportion to C than B hath to C, or if C hath a greater proportion to B than it hath to A; in either of these cases A must be greater than B.*

For was  $A$  equal to, or less than,  $B$ , then either  $A$  must have the same proportion to  $C$  that  $B$  hath to  $C$ , as in the seventh proposition, or a less as in the eighth, both which contradict the first supposition: and again, was  $A$  equal to, or less than,  $B$ , then either  $C$  must have the same proportion to  $A$  that it hath to  $B$ , as in the seventh proposition, or a greater as in the eighth, both which contradict the second supposition; therefore  $A$  must be greater than  $B$ . Q. E. D.

## PROPOSITION 11.

276. *If two ratios be the same with a third, they must be the same with one another: as if the ratio of A to a and the ratio of C to c be both the same with the ratio of B to b, then the ratio of A to a will be the same with the ratio of C to c: or thus; If A be to a as B to b, and B to b as C to c; I say then that A will be to a as C to c.*

For taking any equimultiples of the antecedents, suppose  $3A$ ,  $3B$ ,  $3C$ ; and any other equimultiples of the consequents, suppose  $2a$ ,  $2b$ ,  $2c$ , let  $3A$  be greater than  $2a$ ; then since by the supposition  $A$  is to  $a$  as  $B$  to  $b$ , and  $3A$  is greater than  $2a$ ,  $3B$  must be greater than  $2b$  by the fifth definition: again, since  $B$  is to  $b$  as  $C$  to  $c$ , and  $3B$  is greater than  $2b$ ,  $3C$  must be greater than  $2c$ : thus then we see that if  $3A$  be greater than  $2a$ ,  $3C$  must necessarily be greater than  $2c$ : and in like manner it may be demonstrated that if  $3A$  be equal to, or less than,  $2a$ ,  $3C$  will accordingly be equal to, or less than,  $2c$ . Since then we have four quantities  $A$ ,  $a$ ,  $C$ , and  $c$ , whereof  $3A$  and  $3C$  represent any equimultiples of the first and third, and

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 and  $2a$  and  $2c$  any other equimultiples of the second  
 and fourth; and since  $3A$  cannot be greater than,  
 equal to, or less than,  $2a$ , but  $3C$  must accordingly  
 be greater than, equal to, or less than,  $2c$ , it follows  
 from the fifth definition that these four quantities  
 $A, a, C$ , and  $c$ , must be proportionable,  $A$  to  $a$  as  $C$  to  $c$ .  
 Q. E. D.

# PROPOSITION 12.

277. *If ever so many quantities  $A, B, C$ , in one series  
 be proportionable to as many  $a, b, c$ , in another, that  
 is,  $A$  to  $a$  as  $B$  to  $b$  as  $C$  to  $c$ ; I say then, that as  
 any one antecedent is to its consequent, so will the  
 sum of all the antecedents be to the sum of all the con-  
 sequents; that is, as  $A$  is to  $a$  so will  $\overline{A+B+C}$   
 be to  $\overline{a+b+c}$ : or if we suppose  $A+B+C=S$ , and  
 $a+b+c=s$ , I say then that as  $A$  is to  $a$  so will  
 $S$  be to  $s$ .*

For taking any equimultiples of the antecedents,  
 suppose  $3A, 3B, 3C$ , and any other equimultiples of  
 the consequents, suppose  $2a, 2b, 2c$ , let  $3A$  be greater  
 than  $2a$ ; then since  $A$  is to  $a$  as  $B$  to  $b$ , and  $3A$  is  
 greater than  $2a$ ,  $3B$  must be greater than  $2b$  by the  
 fifth definition: again, since  $B$  is to  $b$  as  $C$  to  $c$ , and  
 $3B$  is greater than  $2b$ ,  $3C$  must be greater than  $2c$ :  
 therefore if  $3A$  be greater than  $2a$ , not only  $3B$  will  
 be greater than  $2b$ , but also  $3C$  will be greater than  
 $2c$ , and consequently the whole sum  $\overline{3A+3B+3C}$   
 will be greater than the whole sum  $\overline{2a+2b+2c}$ :  
 but by the first proposition, the sum  $\overline{3A+3B+3C}$   
 is the same multiple of the sum  $\overline{A+B+C}$  or  $S$  that  
 $3A$  is of  $A$ ; therefore  $\overline{3A+3B+3C}=3S$ ; and for  
 the same reason  $\overline{2a+2b+2c}=2s$ ; therefore we may  
 now say that if  $3A$  be greater than  $2a$ ,  $3S$  will be  
 greater than  $2s$ : and after the same manner might it  
 be demonstrated, that if  $3A$  be equal to, or less than,

$2a$ ,  $3S$  will be equal to, or less than,  $2s$ . Since then we have four quantities  $A$ ,  $a$ ,  $S$ , and  $s$ , whereof  $3A$  and  $3S$  represent any equimultiples of the first and third, and  $2a$  and  $2s$  any others of the second and fourth; and since  $3A$  cannot be greater than, equal to, or less than,  $2a$ , but  $3S$  must in like manner be greater than, equal to, or less than,  $2s$ , it follows from the fifth definition that these four quantities  $A$ ,  $a$ ,  $S$ , and  $s$ , must be proportionable,  $A$  to  $a$  as  $S$  to  $s$ . *Q. E. D.*

### PROPOSITION 13.

278. *If A hath the same proportion to a that B hath to b, but B hath a greater proportion to b than C hath to c; I say then that A hath a greater proportion to a than C to c.*

For since by the supposition  $B$  is to  $b$  in a greater proportion than  $C$  to  $c$ , it follows from the seventh definition that there are equimultiples of  $B$  and  $C$ , and others again of  $b$  and  $c$ , of such a nature that  $B$ 's multiple shall exceed that of  $b$ , and at the same time  $C$ 's multiple shall not exceed that of  $c$ : let then  $3B$  exceed  $2b$ , and let  $3C$  not exceed  $2c$ ; then since  $A$  is to  $a$  as  $B$  to  $b$ , and  $3b$  exceeds  $2b$ ,  $3A$  must necessarily exceed  $2a$  by the fifth definition; therefore we have four quantities  $A$ ,  $a$ ,  $C$ , and  $c$ , whereof  $3A$  and  $3C$  are equimultiples of the first and third, and  $2a$  and  $2c$  are other equimultiples of the second and fourth; and since  $3A$  exceeds  $2a$  when  $3C$  does not exceed  $2c$ , it follows from the seventh definition that of these four quantities  $A$ ,  $a$ ,  $C$ , and  $c$ ,  $A$  hath a greater proportion to  $a$  than  $C$  hath to  $c$ . *Q. E. D.*

### PROPOSITION 14.

279. *If four homogeneous quantities be proportionable, the first to the second as the third to the fourth; I say then that the second will be greater than, equal to, or less than, the fourth, according as the first is greater than,*

than, equal to, or less than, the third: as if  $A$  be to  $B$  as  $C$  is to  $D$ ; I say then that  $B$  will be greater than, equal to, or less than,  $D$ , according as  $A$  is greater than, equal to, or less than,  $C$ .

### C A S E 1.

Let  $A$  be greater than  $C$ : I say then that  $B$  will be greater than  $D$ . For since  $A$  is greater than  $C$ ,  $A$  will have a greater proportion to  $B$  than  $C$  hath to  $B$  by the eighth proposition: again, since  $C$  is to  $D$  as  $A$  to  $B$ , and  $A$  hath a greater proportion to  $B$  than  $C$  hath to  $B$ , it follows from the last proposition that  $C$  is to  $D$  in a greater proportion than  $C$  to  $B$ ; therefore by the tenth proposition  $B$  is greater than  $D$ . *Q. E. D.*

### C A S E 2.

Let now  $A$  be less than  $C$ : I say then that  $B$  will be less than  $D$ . For if  $A$  be less than  $C$ ; then  $C$  will be greater than  $A$ : since then  $C$  is to  $D$  as  $A$  is to  $B$  *ex hypothesi*, and  $C$  is greater than  $A$ , it follows from the last case that  $D$  will be greater than  $B$ ; and therefore  $B$  will be less than  $D$ . *Q. E. D.*

### C A S E 3.

Lastly, let  $A$  be equal to  $C$ : I say then that  $B$  will be equal to  $D$ . For since  $A$  is equal to  $C$ ,  $A$  will be to  $B$  as  $C$  is to  $B$  by the seventh proposition; but  $C$  is to  $D$  as  $A$  to  $B$  by the supposition; therefore  $C$  is to  $D$  as  $C$  is to  $B$  by the eleventh proposition; therefore  $B$  and  $D$  are equal by the ninth. *Q. E. D.*

### P R O P O S I T I O N 15.

280. *Parts are in the same proportion with their respective equimultiples. Let  $A$  and  $a$  be any two homogeneous quantities, whereof  $3A$  and  $3a$  represent any equimultiples respectively; I say then, that  $A$  will be to  $a$  as  $3A$  to  $3a$ .*

For

For take  $B$  and  $C$  both equal to  $A$ , and also  $b$  and  $c$  both equal to  $a$ ; then by the seventh proposition we shall have  $A$  to  $a$  as  $B$  to  $b$  as  $C$  to  $c$ ; therefore by the twelfth proposition we shall have  $A$  to  $a$  as  $\overline{A+B+C}$  to  $\overline{a+b+c}$ : but in this case  $\overline{A+B+C}=3A$ , and  $\overline{a+b+c}=3a$ ; therefore  $A$  is to  $a$  as  $3A$  is to  $3a$ .  
 Q. E. D.

### PROPOSITION 16.

281. *If four homogeneous quantities be proportionable, the first to the second as the third to the fourth; I say then that they will also be alternately proportionable, that is, the first to the third as the second to the fourth: as if  $A$  be to  $B$  as  $C$  to  $D$ ; I say then that  $A$  will be to  $C$  as  $B$  to  $D$ .*

For, taking any equimultiples of  $A$  and  $B$ , suppose  $3A$  and  $3B$ , and any others of  $C$  and  $D$ , suppose  $2C$  and  $2D$ ; since  $3A$  is to  $3B$  as  $A$  to  $B$  by the last, and  $A$  is to  $B$  as  $C$  to  $D$  by the supposition, and  $C$  is to  $D$  as  $2C$  to  $2D$  by the last; it follows from the 11th proposition that  $3A$  is to  $3B$  as  $2C$  to  $2D$ ; therefore by the 14th proposition,  $3A$  cannot be greater than, equal to, or less than,  $2C$ , but at the same time  $3B$  must be greater than, equal to, or less than,  $2D$ . Since then we have four quantities  $A$ ,  $C$ ,  $B$ , and  $D$ , whereof  $3A$  and  $3B$  represent any equimultiples of the first and third, and  $2C$  and  $2D$  any other equimultiples of the second and fourth; and since  $3A$  cannot be greater than, equal to, or less than,  $2C$ , but  $3B$  must accordingly be greater than, equal to, or less than,  $2D$ , it follows from the fifth definition that these four quantities  $A$ ,  $C$ ,  $B$ , and  $D$ , must be proportionable,  $A$  to  $C$  as  $B$  to  $D$ .  
 Q. E. D.

*Note,* Alternate proportion can have no place, except where all the quantities  $A$ ,  $B$ ,  $C$ , and  $D$ , are of the same kind: for if  $A$  and  $B$  were of one kind, and  $C$  and  $D$  of another, how would it be possible for the  
 quantities

quantities  $A$  and  $C$ , or  $B$  and  $D$ , to have any proportion one to another, much less the same?

PROPOSITION 17.

282. *If four quantities  $A$ ,  $B$ ,  $C$ , and  $D$ , whereof  $A$  is greater than  $B$ , and  $C$  greater than  $D$ , be proportionable,  $A$  to  $B$  as  $C$  to  $D$ ; I say then that  $A - B$  will be to  $B$  as  $C - D$  is to  $D$ , which is called proportion by division.*

For let  $3A$ ,  $3B$ ,  $3C$ , and  $3D$ , be any equimultiples of the quantities  $A$ ,  $B$ ,  $C$ , and  $D$ ; then will  $3A - 3B$  and  $3C - 3D$  be like multiples of  $A - B$  and  $C - D$ . Again, let  $2B$  and  $2D$  be any other equimultiples of  $B$  and  $D$ , and let  $3A - 3B$  be greater than  $2B$ ; then if  $3B$  be added to both sides, we shall have  $3A$  greater than  $5B$ ; and because  $A$  is to  $B$  as  $C$  is to  $D$ , we shall have, by the fifth definition,  $3C$  greater than  $5D$ ; take  $3D$  from both sides, and you will have  $3C - 3D$  greater than  $2D$ ; therefore if  $3A - 3B$  be greater than  $2B$ ,  $3C - 3D$  must be greater than  $2D$ ; and by a like process it may be demonstrated, that if  $3A - 3B$  be equal to, or less than,  $2B$ ,  $3C - 3D$  will be equal to, or less than,  $2D$ . Since then we have four quantities,  $A - B$ ,  $B$ ,  $C - D$ , and  $D$ , whereof  $3A - 3B$  and  $3C - 3D$  represent any equimultiples of the first and third, and  $2B$  and  $2D$  any other equimultiples of the second and fourth; and since  $3A - 3B$  cannot be greater than, equal to, or less than  $2B$ , but at the same time  $3C - 3D$  must accordingly be greater than, equal to, or less than,  $2D$ , it follows from the fifth definition that these four quantities,  $A - B$ ,  $B$ ,  $C - D$ , and  $D$ , must be proportionable,  $A - B$  to  $B$  as  $C - D$  to  $D$ . Q. E. D.

## PROPOSITION 18.

283. *If four quantities A, B, C, and D, be proportionable, A to B as C to D; I say then that  $A+B$  will be to B as  $C+D$  to D, which is called proportion by composition.*

If this be denied, that  $A+B$  is to  $B$  as  $C+D$  is to  $D$ , it must then be allowed that  $A+B$  is to  $B$  as  $C+D$  is to some quantity either greater or less than  $D$ ; suppose to a greater, and call it  $E$ ; then since  $E$  is by the supposition greater than  $D$ , if  $C-E$  be added to both sides, we shall have  $C$  greater than  $C+D-E$ . This being observed, let us begin again, and suppose  $A+B$  to  $B$  as  $C+D$  to  $E$ ; then we shall have *dividendo* (that is, by the last proposition)  $A+B-B$  to  $B$  as  $C+D-E$  to  $E$ ; but  $A+B-B$  is equal to  $A$ ; therefore  $A$  is to  $B$  as  $C+D-E$  is to  $E$ ; but  $A$  is to  $B$  as  $C$  is to  $D$  by the supposition; therefore  $C$  is to  $D$  as  $C+D-E$  is to  $E$ ; but of these four proportionals  $C$ ,  $D$ ,  $C+D-E$ , and  $E$ , it has been proved that the first is greater than the third, that  $C$  is greater than  $C+D-E$ ; therefore, by the fourteenth, the second must be greater than the fourth, that is,  $D$  must be greater than  $E$ ; therefore  $E$  must be less than  $D$ ; therefore if  $A+B$  be to  $B$  as  $C+D$  is to any quantity greater than  $D$ , that quantity must also be less than  $D$ , which is impossible; therefore it is impossible for  $A+B$  to be to  $B$  as  $C+D$  is to any quantity greater than  $D$ : and by a like process it may be demonstrated, that it is as impossible for  $A+B$  to be to  $B$  as  $C+D$  is to any quantity less than  $D$ ; therefore  $A+B$  must be to  $B$  as  $C+D$  is to  $D$ .  
Q. E. D.

PROPOSITION 19.

284. *If from two quantities A and B in any proportion be subtracted other two C and D in the same proportion; I say then that the remainders  $A - C$  and  $B - D$  will still be in the same proportion, that is, that  $A - C$  will be to  $B - D$  as A to B or as C to D.*

For since by the supposition  $A$  is to  $B$  as  $C$  is to  $D$ , we shall have *permutando* (that is, by the sixteenth proposition)  $A$  to  $C$  as  $B$  to  $D$ ; and *dividendo*,  $A - C$  to  $C$  as  $B - D$  to  $D$ ; and again *permutando*,  $A - C$  to  $B - D$  as  $C$  is to  $D$ ; but  $A$  is to  $B$  as  $C$  is to  $D$ ; therefore  $A - C$  is to  $B - D$  as  $A$  to  $B$ . Q. E. D.

SCHOLIUM.

Here Doctor Gregory in his manuscript copy finds a corollary demonstrating that illation called conversion of proportion; but because it is difficult to make sense of that demonstration, I chuse rather to insert his own demonstration of the same proposition, which is as follows:

*If four quantities A, B, C, and D, be proportionable, A to B as C to D; I say then that A is to  $A - B$  as C is to  $C - D$ , which is called conversion of proportion.* For since by the supposition  $A$  is to  $B$  as  $C$  is to  $D$ , we shall have *dividendo*,  $A - B$  to  $B$  as  $C - D$  to  $D$ ; and *invertendo*,  $B$  to  $A - B$  as  $D$  to  $C - D$ ; and *componendo*,  $B + A - B$  to  $A - B$  as  $D + C - D$  to  $C - D$ , that is,  $A$  to  $A - B$  as  $C$  to  $C - D$ . Q. E. D.

As to the foregoing nineteenth proposition I shall further observe, that as in that proposition, by division of proportion it was demonstrated, that if from two quantities  $A$  and  $B$  in any proportion be subtracted two others  $C$  and  $D$  in the same proportion, the remainders  $A - C$  and  $B - D$  will still be in the same proportion with  $A$  and  $B$ ; so by composition of proportion it may be demonstrated, that if to two quantities  $A$  and  $B$  in any proportion be added

two others  $C$  and  $D$  in the same proportion, the aggregates  $A + C$  and  $B + D$  will still be in the same proportion with  $A$  and  $B$ ; but this has already been demonstrated, being a particular case of the twelfth proposition.

### PROPOSITION 20.

285. *If there be three quantities  $A$ ,  $B$ , and  $C$ , in one series, and three others  $D$ ,  $E$ , and  $F$ , in another, and if the proportions in one series be the same with the proportions in the other when taken in the same order, as if  $A$  be to  $B$  as  $D$  is to  $E$ , and  $B$  to  $C$  as  $E$  to  $F$ ; I say then that  $A$  cannot be greater than, equal to, or less than,  $C$  in one series, but accordingly  $D$  must be greater than, equal to, or less than,  $F$  in the other.*

For let  $A$  be greater than  $C$ ; then it is plain from the eighth proposition that  $A$  must have a greater proportion to  $B$  than  $C$  hath to  $B$ ; but  $A$  is to  $B$  as  $D$  to  $E$  by the supposition, and  $C$  is to  $B$  as  $F$  to  $E$ , because by the supposition  $B$  is to  $C$  as  $E$  to  $F$ ; therefore  $D$  hath a greater proportion to  $E$  than  $F$  hath to  $E$ ; therefore  $D$  is greater than  $F$  by the tenth proposition; therefore if  $A$  be greater than  $C$ ,  $D$  must be greater than  $F$ : and after the same manner it may be demonstrated, that if  $A$  be equal to, or less than,  $C$ ,  $D$  must accordingly be equal to, or less than,  $F$ : therefore  $A$  cannot be greater than, equal to, or less than,  $C$ , but accordingly  $D$  must be greater than, equal to, or less than,  $F$ . Q. E. D.

### PROPOSITION 21.

286. *If there be three quantities  $A$ ,  $B$ , and  $C$ , in one series, and three others,  $D$ ,  $E$ , and  $F$ , in another, and if the proportions in one series be the same with the proportions in the other, but in a different order, as if  $A$  be to  $B$  as  $E$  is to  $F$ , and  $B$  to  $C$  as  $D$  is to  $E$ ; I say still that  $A$  cannot be greater than, equal to, or less than,  $C$ , but accordingly  $D$  must be greater than, equal to, or less than,  $F$ .*

For

For let  $A$  be greater than  $C$ ; then by the eighth proposition  $A$  must have a greater proportion to  $B$  than  $C$  hath to  $B$ : but  $A$  is to  $B$  as  $E$  is to  $F$  by the supposition, and  $C$  is to  $B$  as  $E$  to  $D$ , because by the supposition  $B$  is to  $C$  as  $D$  to  $E$ ; therefore  $E$  hath a greater proportion to  $F$  than it hath to  $D$ ; therefore  $D$  must be greater than  $F$  by the tenth proposition; therefore if  $A$  be greater than  $C$ ,  $D$  must be greater than  $F$ : and by a like way of reasoning, if  $A$  be equal to, or less than,  $C$ ,  $D$  will accordingly be equal to, or less than,  $F$ ; therefore  $A$  cannot be greater than, equal to, or less than,  $C$ , but accordingly  $D$  must be greater than, equal to, or less than,  $F$ . *Q. E. D.*

PROPOSITION 22.

287. *If there be three quantities  $A$ ,  $B$ , and  $C$ , in one series, and three others  $D$ ,  $E$ , and  $F$ , in another, and if the proportions in one series be the same with the proportions in the other when taken in the same order; I say then that the extremes in one series will be in the same proportion with the extremes in the other: as if  $A$  be to  $B$  as  $D$  is to  $E$ , and  $B$  to  $C$  as  $E$  to  $F$ ; I say then that  $A$  will be to  $C$  as  $D$  to  $F$ .*

*Note,* For avoiding a multiplicity of words, this consequence is said to follow *ex æquo ordinate*, or *ex æquo*: see the eighteenth and nineteenth definitions.

Take any equimultiples of  $A$  and  $D$ , suppose  $4A$  and  $4D$ , and any others of  $B$  and  $E$ , suppose  $3B$  and  $3E$ , and lastly any others of  $C$  and  $F$ , as  $2C$  and  $2F$ ; then since by the supposition  $A$  is to  $B$  as  $D$  is to  $E$ , it follows from the fourth proposition that  $4A$  will be to  $3B$  as  $4D$  to  $3E$ : again, since by the supposition  $B$  is to  $C$  as  $E$  to  $F$ , it follows from the same fourth proposition that  $3B$  will be to  $2C$  as  $3E$  to  $2F$ : so that we have three quantities, to wit,  $4A$ ,  $3B$ ,  $2C$ , in one series, and three others, to wit,  $4D$ ,  $3E$ , and  $2F$ , in another; and it has been shewn that the proportions in one series are the same with the proportions

in the other when taken in the same order, that is,  $4A$  is to  $3B$  as  $4D$  to  $3E$ , and  $3B$  to  $2C$  as  $3E$  to  $2F$ ; therefore, by the twentieth proposition,  $4A$  cannot be greater than, equal to, or less than,  $2C$ , but  $4D$  must accordingly be greater than, equal to, or less than,  $2F$ . Since then we have four quantities  $A$ ,  $C$ ,  $D$ , and  $F$ , whereof  $4A$  and  $4D$  represent any equimultiples of the first and third, and  $2C$  and  $2F$  any other equimultiples of the second and fourth; and since  $4A$  cannot be greater than, equal to, or less than,  $2C$ , but accordingly  $4D$  must be greater than, equal to, or less than,  $2F$ , it follows from the fifth definition that these four quantities  $A$ ,  $C$ ,  $D$ , and  $F$ , are proportionable,  $A$  to  $C$  as  $D$  to  $F$ . Q. E. D.

## C O R O L L A R Y.

*In like manner, if there be ever so many quantities  $A$ ,  $B$ ,  $C$ ,  $G$ , &c. in one series, and as many others  $D$ ,  $E$ ,  $F$ ,  $H$ , &c. in another, and if  $A$  be to  $B$  as  $D$  is to  $E$ , and  $B$  to  $C$  as  $E$  to  $F$ , and  $C$  to  $G$  as  $F$  to  $H$ , &c. the consequence with respect to the extremes will still be the same, that is,  $A$  will be to  $G$  as  $D$  to  $H$ : for it has been proved already that  $A$  is to  $C$  as  $D$  to  $F$ ; and by the supposition  $C$  is to  $G$  as  $F$  to  $H$ ; therefore, ex æquo,  $A$  will be to  $G$  as  $D$  to  $H$ .*

## P R O P O S I T I O N 23.

288. *If there be three quantities  $A$ ,  $B$ , and  $C$ , in one series, and three others  $D$ ,  $E$ , and  $F$ , in another, and if the proportions in one series be the same with the proportions in the other, but in a different order; I say that the extremes in one series will still be in the same proportion with the extremes in the other: as if  $A$  be to  $B$  as  $E$  is to  $F$ , and  $B$  to  $C$  as  $D$  to  $E$ ; I say still that  $A$  will be to  $C$  as  $D$  to  $F$ .*

*Note.* This consequence is said to be *ex æquo perturbate*.

Take any equimultiples of  $A$ ,  $B$ , and  $D$ , suppose  $3A$ ,  $3B$ , and  $3D$ , and any others of  $C$ ,  $E$ , and  $F$ , suppose

pose  $2C$ ,  $2E$ , and  $2F$ , and the reasoning is as follows:  $3A$  is to  $3B$  as  $A$  to  $B$  by the fifteenth, and  $A$  is to  $B$  as  $E$  to  $F$  by the supposition, and  $E$  is to  $F$  as  $2E$  to  $2F$  by the fifteenth; therefore  $3A$  is to  $3B$  as  $2E$  is to  $2F$  by the eleventh; again,  $B$  is to  $C$  as  $D$  to  $E$  by the supposition; therefore  $3B$  will be to  $2C$  as  $3D$  to  $2E$  by the fourth: since then we have three quantities, to wit,  $3A$ ,  $3B$ , and  $2C$ , in one series, and three others, to wit,  $3D$ ,  $2E$ , and  $2F$ , in another, and since the proportions are the same in both serieses, but in a different order, that is, since  $3A$  is to  $3B$  as  $2E$  to  $2F$ , and  $3B$  is to  $2C$  as  $3D$  to  $2E$ , it follows from the twenty-first proposition, that  $3A$  cannot be greater than, equal to, or less than,  $2C$ , but  $3D$  must accordingly be greater than, equal to, or less than,  $2F$ : again, since we have four quantities  $A$ ,  $C$ ,  $D$ , and  $F$ , whereof  $3A$  and  $3D$  represent any equimultiples of the first and third, and  $2C$  and  $2F$  any others of the second and fourth, and since  $3A$  cannot be greater than, equal to, or less than,  $2C$ , but  $3D$  must accordingly be greater than, equal to, or less than  $2F$ , it follows from the fifth definition that these four quantities  $A$ ,  $C$ ,  $D$ , and  $F$ , are proportionable,  $A$  to  $C$  as  $D$  to  $F$ .  $\mathcal{Q}. E. D.$

PROPOSITION 24.

289. *If there be six quantities,  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ , whereof  $A$  is to  $B$  as  $C$  is to  $D$ , and  $E$  is to  $B$  as  $F$  to  $D$ ; I say then that  $A+E$  will be to  $B$  as  $C+F$  to  $D$ .*

For since by the supposition  $E$  is to  $B$  as  $F$  to  $D$ , we shall have, *invertendo*,  $B$  to  $E$  as  $D$  to  $F$ . Since then  $A$  is to  $B$  as  $C$  is to  $E$  by the supposition, and  $B$  is to  $E$  as  $D$  to  $F$ , it follows *ex æquo*, that  $A$  is to  $E$  as  $C$  to  $F$ ; whence, *componendo*,  $A+E$  will be to  $E$  as  $C+F$  is to  $F$ : again, since  $A+E$  is to  $E$  as  $C+F$  is to  $F$ , and  $E$  is to  $B$  as  $F$  to  $D$  by the supposition, it follows again *ex æquo*, that  $A+E$  is to  $B$  as  $C+F$  to  $D$ .  $\mathcal{Q}. E. D.$

## L E M M A.

290. *If four quantities A, B, C, and D, be proportionable, A to B as C to D; I say then that A cannot possibly be greater than, equal to, or less than, B, but that C will accordingly be greater than, equal to, or less than, D.*

That this lemma is self-evident according to the common notion of proportionality, or even upon the plan of the fifth definition, were simple quantities allowed to be considered as equimultiples of themselves, is what I suppose will scarcely be denied: but this the name of multiple and equimultiple will by no means admit of, and therefore care has been taken to provide against it, as may be seen in my observations on the second definition, and at the end of the sixth proposition: therefore, as the doctrine of proportion here stands, this lemma ought certainly to be demonstrated; and the author's taking it for granted in the demonstration of the next proposition following, where he might with so much ease have avoided it, is not so much an argument of its self-evidency, as that he had demonstrated it somewhere before in this fifth book, but that it is now lost. *Commandine*, from the fourteenth of this book, has demonstrated one particular case of this proposition, that is, where the quantities *A*, *B*, *C*, and *D*, are all of a kind: but this proposition is no less true when the quantities *A* and *B* are of one kind, and *C* and *D* of another. This *Clavius* very well observes, and endeavours to demonstrate this proposition in this more extended sense (see his scholium to the fourteenth proposition of the fifth book); but whether this demonstration of his amounts to any more than proving *idem per idem*, let them that read it judge. The demonstration I shall here give of it is as follows:

I am to demonstrate that if *A* be to *B* as *C* is to *D*; then *A* cannot possibly be greater than, equal to,  
or

Art. 290, 291. *The fifth Book of EUCLID's Elements.* 313  
 or less than,  $B$ , but accordingly  $C$  must be greater than,  
 equal to, or less than,  $D$ .

### C A S E 1.

Let  $A$  be greater than  $B$ ; I say then that  $C$  must be greater than  $D$ . For since  $A$  is greater than  $B$ , multiply the excess  $A - B$  to a multiple greater than  $B$ , and let this multiple be  $3A - 3B$ ; then since  $3A - 3B$  is greater than  $B$ , if  $3B$  be added to both sides, we shall have  $3A$  greater than  $4B$ : again, since  $A$  is to  $B$  as  $C$  is to  $D$ , and  $3A$  is greater than  $4B$ , we shall have, by the fifth definition,  $3C$  greater than  $4D$ ; therefore  $3C$  must be much greater than  $3D$ , and  $C$  must be greater than  $D$ . Q. E. D.

### C A S E 2.

Let now  $A$  be less than  $B$ ; I say then that  $C$  must be less than  $D$ . For since  $A$  is to  $B$  as  $C$  is to  $D$ , we shall have, *invertendo*,  $B$  to  $A$  as  $D$  to  $C$ ; but  $B$  is greater than  $A$ , because by the supposition  $A$  is less than  $B$ ; therefore  $D$  must be greater than  $C$  by the last case; therefore  $C$  must be less than  $D$ . Q. E. D.

### C A S E 3.

Lastly, let  $A$  be equal to  $B$ ; I say then that  $C$  must be equal to  $D$ . For since  $C$  is to  $D$  as  $A$  is to  $B$ , should  $C$  be greater or less than  $D$ ,  $A$  would accordingly be greater or less than  $B$  by the two last cases; but  $A$  is neither greater nor less than  $B$  by the supposition; therefore  $C$  is neither greater nor less than  $D$ ; therefore  $C$  is equal to  $D$ . Q. E. D.

### P R O P O S I T I O N 25.

291. *If four quantities A, B, C, and D, be proportionable, A to B as C to D; I say then that the sum of the greatest and least terms put together will be greater than the sum of the other two.*

Let

Let  $A$  be the greatest of all; then, since  $A$  is to  $B$  as  $C$  is to  $D$ , and  $B$  is less than  $A$ ,  $D$  will be less than  $C$  by the lemma: again, since  $A$  is to  $B$  as  $C$  is to  $D$ , and  $C$  is less than  $A$ ,  $D$  will be less than  $B$  by the fourteenth; therefore if  $A$  be the greatest of all,  $D$ , which is less than either  $A$ ,  $B$ , or  $C$ , will be the least of all, and so the sum of the greatest and least terms added together will be  $A + D$ ; therefore the sum of the other two will be  $B + C$ . We are now then to prove that the sum  $A + D$  is greater than the sum  $B + C$ , which is thus done: It has been demonstrated in the nineteenth proposition, that if from two quantities  $A$  and  $B$  in any proportion whatever, be subtracted other two  $C$  and  $D$  in the same proportion, the remainder  $A - C$  will be to the remainder  $B - D$  as  $A$  to  $B$ ; but  $A$  is greater than  $B$  by the supposition; therefore  $A - C$  must be greater than  $B - D$  by the lemma; add  $C + D$  to both sides, and you will have  $A + D$  greater than  $B + C$ . Q. E. D.

#### C O R O L L A R Y.

*If three quantities  $A$ ,  $B$ , and  $C$ , be in continual proportion,  $A$  to  $B$  as  $B$  to  $C$ ; I say then that the sum of the extremes will be greater than twice the middle term, that  $A + C$  will be greater  $B + B$  or  $2B$ .*

#### Of the COMPOSITION and RESOLUTION of RATIOS.

*N. B.* As numbers are quantities whereof we have more distinct ideas than of any other quantities whatever, and as all ratios must be reduced to those of numbers before we can make any considerable use of their composition and resolution in computing the quantities of time, space, velocity, motion, force, &c. I shall confine myself chiefly to this sort of ratios in what I have to deliver in the following articles.

## DEFINITION I.

292. *In comparing ratios, that ratio is said to be greater than, equal to, or less than, another, whose antecedent hath a greater, or an equal, or a less proportion to its consequent than the other's antecedent hath to its consequent.* Thus the ratio of 6 to 3 is said to be greater, and the ratio of 4 to 3 less than the ratio of 5 to 3; thus again the ratio of 6 to 3 is said to be greater, and the ratio of 6 to 5 less than the ratio of 6 to 4, &c. *Therefore whenever two ratios are to be compared whose antecedents and consequents are both different, it will be proper to reduce them to the same antecedent or to the same consequent before the comparison is made.* As for instance; suppose any one would know which of these two ratios is the greater, to wit, the ratio of 7 to 5, or the ratio of 4 to 3: to know this, it will be proper to set off one of the ratios: suppose that of 4 to 3, from 7 the antecedent of the other (by which phrase I mean no more than finding a number to which 7 hath the same proportion that 4 hath to 3); and this may be done by saying, as 4 is to 3, so is 7 to  $\frac{21}{4}$ , or  $5\frac{1}{4}$ : thus then it appears that the proportion of 4 to 3 is the same with the proportion of 7 to  $5\frac{1}{4}$ ; so that now the question turns upon this, which of these two ratios is the greater, that of 7 to 5, or that of 7 to  $5\frac{1}{4}$ ? and the answer is ready, to wit, that the ratio of 7 to 5 is the greater ratio, by the eighth proposition of the fifth book of the Elements; therefore the ratio of 7 to 5 is greater than the ratio of 4 to 3. Again, suppose I would compare the ratio of 3 to 4 with the ratio of 5 to 7; then I would set off the ratio of 3 to 4 from 5, by saying, as 3 is to 4, so is 5 to  $\frac{20}{3}$ , or  $7 - \frac{1}{3}$ ; whereby it appears that the ratio of 3 to 4 is the same with the ratio of 5 to  $7 - \frac{1}{3}$ ; but the proportion of 5 to  $7 - \frac{1}{3}$  is greater than the  
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proportion of 5 to 7, as is evident from the eighth proposition of the fifth book of the Elements, and also from the very nature of ratios, the number 5 having more magnitude when compared with  $7 - \frac{2}{7}$  than it hath when compared with 7; therefore the ratio of 3 to 4 is greater than the ratio of 5 to 7.

There is also another way of comparing ratios, by turning their terms into fractions, making the antecedents numerators, and the consequents denominators. Thus the ratio of  $A$  to  $B$  is greater than, equal to, or less than, the ratio of  $C$  to  $D$ , according as the fraction  $\frac{A}{B}$  is greater than, equal to, or less

than, the fraction  $\frac{C}{D}$ : for the ratio of  $\frac{A}{B}$  to 1 is greater than, equal to, or less than, the ratio of  $\frac{C}{D}$  to 1, according as the fraction  $\frac{A}{B}$  is greater than, equal to, or less than, the fraction  $\frac{C}{D}$ ; this is evident

from what has been laid down already: but the ratio of  $\frac{A}{B}$  to 1 is the same with the ratio of  $A$  to  $B$ , and

the ratio of  $\frac{C}{D}$  to 1 is the same with the ratio of  $C$  to  $D$ ; therefore the ratio of  $A$  to  $B$  is greater than, equal to, or less than, the ratio of  $C$  to  $D$ , according as the fraction  $\frac{A}{B}$  is greater than, equal to, or less

than, the fraction  $\frac{C}{D}$ . But this way of representing ratios by fractions, though it may serve well enough for comparing them as to greater and less, yet it ought not by any means to be admitted in general, because these representatives are not in the same proportion with the ratios represented by them: thus the fraction  $\frac{6}{2}$  is double of the fraction  $\frac{3}{2}$ , but yet it must  
by

by no means be concluded from thence that the ratio of 6 to 2 is double of the ratio of 3 to 2; for it will be found hereafter that the ratio of 9 to 4 is double of the ratio of 3 to 2. For my own part, I never was a favourer of representing ratios by fractions, or even fraction-wise, as is done by *Barrow* and others; not only for the reasons above given, but also because that this way of representing ratios is very apt to mislead beginners into wrong conceptions of their composition and resolution.

### DEFINITION 2.

293. *In a series of quantities of any kind whatsoever increasing or decreasing from the first to the last, the ratio of the extremes is said to be compounded of all the intermediate ratios. Thus if  $A, B, C, D$ , represent any number of quantities put down (or imagined to be put down)  $A, B, C, D$ , in a series, the ratio of  $A$  to  $D$  is 48, 40, 30, 15, said to be compounded of, or to be resolvable into these ratios, to wit, the ratio of  $A$  to  $B$ , the ratio of  $B$  to  $C$ , and the ratio of  $C$  to  $D$ : or thus: If  $A$  and  $D$  be any two quantities, and if  $B, C$ , &c. represent any number of other intermediate quantities interposed at pleasure between  $A$  and  $D$ , the ratio of  $A$  to  $D$  is said by this interposition to be resolved into the ratios of  $A$  to  $B$ , of  $B$  to  $C$ , and of  $C$  to  $D$ .*

This is no proposition to be proved, but a definition laid down of what Mathematicians commonly mean by the composition and resolution of ratios, which is certainly no more than what they mean by composition and resolution in the case of any other *continuum* whatever. As for instance; suppose the letters  $A, B, C, D$ , instead of representing quantities, to represent so many distinct points placed in a right line one after another, whether at equal or unequal distances it matters not: who then would scruple to say that the whole interval  $AD$  consisted of the intervals  $AB, BC, CD$ , as of its parts? Or, if the points  
 $A$  and

*A* and *D* be the extremities of a line, and any number of points *B*, *C*, &c. be marked at pleasure upon it; who will not say that the line *AD* is by these points resolved or distinguished into the parts *AB*, *BC*, *CD*, &c.? This is the case in the composition and resolution of lines; and I see no difference when applied to the composition and resolution of ratios, except that here the whole and all its parts are lines, and there the whole and all its parts are ratios.

If *A*, *B*, *C*, *D*, &c. signify quantities, the ratio of *A* to *B* begins at *A* and terminates in *B*; the ratio of *B* to *C* begins at *B* where the former left off, and terminates in *C*; and the ratio of *C* to *D* begins at *C* and terminates in *D*: why then should not these continued ratios be conceived as parts constituting the whole ratio of *A* to *D*? That ratios are capable of being compared as to greater and less, and that one ratio may be greater than, equal to, or less than, another, we have seen already; and if so, why should not ratios be allowed to have quantity as well as all other things that are capable of being so compared? but if ratios have quantity, they must have parts, and these parts must be of the same nature with the whole, because ratios are not capable of being compared with any thing but ratios: therefore I do not see but that the idea I have here given of the composition and resolution of ratios is as just and as intelligible as it is when applied to any other composition or resolution whatsoever.

To proceed then: let *A*, *B*, *C*, *D*, be points in a right line as before; let the line *AB* be equal to any line *Rr*, let *BC* be equal to some other line *Ss*, and *CD* to the line *Tt*; then it will not only be proper to say that the line *AD* is equal to the three lines *AB*, *BC*, *CD*, but also that the same line *AD* is equal to the three lines *Rr*, *Ss*, and *Tt*, put together: and the same consideration is still applicable to ratios; for supposing *A*, *B*, *C*, *D*, again to signify quantities, as also *R*, *S*, *T*, *r*, *s*, *t*; let *A* be to *B* as *R* to *r*, let *B* be  
to

to  $C$  as  $S$  to  $s$ , and let  $C$  be to  $D$  as  $T$  to  $t$ ; then it is usual amongst Mathematicians not only to consider the ratio of  $A$  to  $D$  as compounded of the lesser ratios of  $A$  to  $B$ , of  $B$  to  $C$ , and of  $C$  to  $D$ , but also as compounded of the ratios of  $R$  to  $r$ , of  $S$  to  $s$ , and of  $T$  to  $t$ . All this will be very intelligible, if we attend to the series already described; for there the ratio of 48 to 15 was compounded of the ratio of 48 to 40, of 40 to 30, and of 30 to 15; but 48 is to 40 as 6 to 5, and 40 is to 30 as 4 to 3, and 30 is to 15 as 2 to 1; therefore it is as proper to consider the ratio of 48 to 15 as compounded of the ratios of 6 to 5, of 4 to 3, and of 2 to 1, as it is to consider it as compounded of the ratios of 48 to 40, of 40 to 30, and of 30 to 15.

### DEFINITION 3.

294. *As when a line is divided into any number of equal parts, the whole line is said to be such a multiple of any one of these parts as is expressed by the number of parts into which the whole is supposed to be divided; so in a series of continual proportionals, where the intermediate ratios are all equal to one another, and consequently to some common ratio that indifferently represents them all, the ratio of the extremes is said to be such a multiple of this common ratio as is expressed by the number of ratios from one extreme to the other.* Thus 9, 6, and 4, are continual proportionals whose common ratio is that of 3 to 2; for 9 is to 6 as 3 to 2, and 6 is to 4 as 3 to 2; therefore, in this case, the ratio of 9 to 4 is said to be the double of the ratio of 3 to 2; and on the other hand, the ratio of 3 to 2 is said to be the half of the ratio of 9 to 4; but the common expression is, that 9 is to 4 in a duplicate ratio of 3 to 2, and 3 is to 2 in a subduplicate ratio of 9 to 4. Again, 27, 18, 12, and 8, are in a continual proportion whose common ratio is that of 3 to 2; therefore 27 is to 8 in a triplicate ratio of 3 to 2, and 3 is to 2 in a subtriplicate ratio of 27 to 8. Lastly, 81, 54, 36, 24, and

and 16 are continual proportionals, whose common ratio is that of 3 to 2; therefore 81 is to 16 in a quadruplicate ratio of 3 to 2, and 3 is to 2 in a subquadruplicate ratio of 81 to 16. By these instances we see that one ratio may not only be greater or less than another, but a multiple, or an aliquot part of another; nay there is no proportion can be assigned which some one ratio may not have to another: thus the ratio of 81 to 16 is found to be to the ratio of 27 to 8, as 4 to 3, because the former ratio contains the ratio of 3 to 2 four times, and the latter three times; thus again, the ratio of 27 to 8 is the ratio of 9 to 4, as 3 to 2, because the former contains the ratio of 3 to 2 three times, and the latter twice; whereby it appears that proportion is compatible even to ratios themselves, as well as to all other continued quantities whatever. But though all ratios are in some certain proportion one to another, yet this proportion cannot always be expressed; I mean, when the quantities of ratios are incommensurable to one another; for ratios may be incommensurable as well as any other continued quantities of what kind soever: thus the ratio of 4 to 3 is incommensurable to the ratio of 3 to 2; which is the case of most ratios, though not of all. If all ratios were commensurable to one another, their logarithms would be so too; and so the logarithms of all the natural numbers might be accurately assigned; whereas from other principles we know to the contrary, as will be seen when we come to treat particularly of logarithms.

*N.B.* The best way of representing the quantities of ratios, that I know of, is by *Gunter's* line, where as many of the natural numbers as can be placed upon it are disposed, not at equal distances one from another, but at distances proportionable to the ratios they are in one to another. Thus the distance between 1 and 2 is equal to the distance between 2 and 4, because the ratio of 1 to 2 is equal to the ratio of

2 to 4 : thus again, the distance between 4 and 9 is double the distance between 2 and 3, because the ratio of 4 to 9 is double the ratio of 2 to 3; and so of the rest.

### *Of the Addition of Ratios.*

295. Since all ratios are quantities, as has been shewn in the three last articles, it follows, that they also as well as all other quantities must be capable of addition, subtraction, multiplication, and division: to treat then of these operations in their order, I shall begin first with addition.

*If the ratios to be added be continued ratios, that is, if they lie in a series wherein the antecedent of every subsequent ratio is the same with the consequent of the ratio that went immediately before, their addition is best performed by throwing out all the intermediate terms: thus the ratios of A to B, of B to C, and of C to D, when added together, make up the ratio of A to D, as was shewn in the 293d article.*

Therefore, if the ratios to be added be discontinued, it will be proper to continue them from some given antecedent, suppose from unity, before they can be added, thus: let the ratio of A to B, the ratio of C to D, and the ratio of E to F, be proposed to be added into one sum: now the ratio of A to B set off from 1 reaches to  $\frac{B}{A}$ , because A is to B as 1 to  $\frac{B}{A}$ ; the next ratio of C to D set off from  $\frac{B}{A}$  reaches to  $\frac{BD}{AC}$ ; and the last ratio of E to F set off from  $\frac{BD}{AC}$  reaches to  $\frac{BDF}{ACE}$ ; therefore the ratios of A to B, of C to D, and of E to F, when added together, make the ratio

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of

of 1 to  $\frac{BDF}{ACE}$ , which is the same with the ratio of  $ACE$  to  $BDF$ ; whence we have the following canon:

*Multiply first the antecedents of all the ratios proposed together, and then their consequents, and the ratio of the products thence arising will be the sum of the ratios proposed.*

That the ratio of  $A$  to  $B$ , of  $C$  to  $D$ , and of  $E$  to  $F$ , all together constitute the ratio of  $ACE$  to  $BDF$ , may be further confirmed by setting them off from  $ACE$  and from one another thus: the ratio of  $A$  to  $B$  set off from  $ACE$  reaches to  $BCE$ ; in the next place the ratio of  $C$  to  $D$  set off from  $BCE$  reaches to  $BDE$ ; and lastly the ratio of  $E$  to  $F$  set off from  $BDE$  reaches to  $BDF$ ; therefore all these ratios together constitute the ratio of  $ACE$  to  $BDF$ . An example in numbers take as follows: let it be required to add these four ratios together, viz. the ratio of 2 to 3, the ratio of 4 to 5, the ratio of 6 to 7, and the ratio of 8 to 9. Here the product of the antecedents is  $2 \times 4 \times 6 \times 8 = 384$ , and the product of the consequents is  $3 \times 5 \times 7 \times 9 = 945$ ; therefore the sum of all the ratios proposed is the ratio of 384 to 945; and the proof is easy: for the ratio of 2 to 3 reaches from 384 to 576; the ratio of 4 to 5 reaches from 576 to 720; the ratio of 6 to 7 reaches from 720 to 840; and the ratio of 8 to 9 reaches from 840 to 945; therefore the ratios of 2 to 3, of 4 to 5, of 6 to 7, and of 8 to 9, reach from 384 to 945.

From what has here been said concerning the addition of ratios, may easily be understood an expression so frequent among Mechanical and Philosophical writers; as when they say that  $A$  is to  $B$  in a ratio compounded of the ratio of  $C$  to  $D$ , and of the ratio of  $E$  to  $F$ ; whereby they mean no more than that the ratio of  $A$  to  $B$  is equal to the sum of the ratios of  $C$  to  $D$ , and of  $E$  to  $F$ ; or that  $A$  is to  $B$  as  $CE$  to  $DF$ .

According to the Mathematicians, every ratio is either a *ratio majoris inaequalitatis*, or a *ratio aequalitatis*,

*tatis*, or a *ratio minoris inæqualitatis*, which takes in all sort of ratios : for by a *ratio majoris inæqualitatis* they mean the ratio that any greater quantity hath to a less ; by a *ratio minoris inæqualitatis* they mean the contrary, that is, the ratio of a lesser quantity to a greater ; and therefore by a *ratio æqualitatis* they mean the ratio (if it may be called so) that every quantity hath to its equal. If we distinguish ratios according to the effects they have in composition, then every *ratio majoris inæqualitatis* ought to be looked upon as affirmative, because such ratios always increase those to which they are added ; on the other hand, the *rationes minoris inæqualitatis* ought to be considered as negative, because these always diminish the ratios to which they are added ; therefore the *ratio æqualitatis* ought to be looked upon as having no magnitude at all, because such ratios have no effect in composition. Thus if to the ratio of 5 to 3 be added the ratio of 3 to 2, the sum will be the ratio of 5 to 2, as above ; but the ratio of 5 to 2 is greater than the ratio of 5 to 3 ; therefore the ratio of 3 to 2 ought to be looked upon as affirmative, because it increases the ratio to which it is added : on the other hand, if the ratio of 5 to 3 be added to the ratio of 3 to 4, the sum will be the ratio of 5 to 4, which is less than the ratio of 5 to 3, and therefore the ratio of 3 to 4 is negative : lastly, if to the ratio of 5 to 3 be added the ratio of 3 to 3, the sum will still be the ratio of 5 to 3 ; therefore the ratio of 3 to 3 is nothing.

Whenever a ratio is to be resolved into two others by any arbitrary interposition of an intermediate term, it may be thought, however, that this intermediate term should be some intermediate magnitude between the terms of the ratio to be resolved ; and so we supposed it in the 293d article : but that restriction was only supposed to prevent unseasonable objections that might otherwise arise about it ; for there is no necessity that the intermediate term should be of an intermediate

magnitude betwixt the extremes, if we allow of negative ratios; for the ratio of 5 to 4 (for instance) may be resolved into the two ratios of 5 to 3 and of 3 to 4, though the intermediate term 3 be out of the limits of 5 to 4. This I say is plain; for though the ratio of 5 to 3, which is one of the parts, be greater than the ratio of 5 to 4, yet the ratio of 3 to 4, which is the other part, is negative, and qualifies the other in the composition, so as to reduce it to the ratio of 5 to 4: so 9 may be looked upon as a part of 7, provided the other part be  $-2$ .

#### C O R O L L A R Y.

*If there be a series of quantities A, B, C, D, whereof A is to B as R to r, and B is to C as S to s, and C is to D as T to t; I say then that A will be to D as RST, the product of all the antecedents, to rst the product of all the consequents.* For by the art. 293, the ratio of A to D is compounded of the ratios of R to r, of S to s, and of T to t; and these ratios, when thrown into one sum, constitute the ratio of RST to rst; therefore A is to D as RST to rst.

#### *Of the Subtraction of Ratios.*

296. *The subtraction of ratios one from another, when both have the same antecedent, or both the same consequent, is obvious enough: thus the ratio of A to B subtracted from the ratio of A to C leaves the ratio of B to C; and the ratio of B to C subtracted from the ratio of A to C leaves the ratio of A to B: this I say is obvious, because (according to art 293) the ratio of A to C contains the ratios of A to B and of B to C; and therefore, if either part be taken away, there must remain the other.*

But if the two ratios, whereof one is to be subtracted from the other, have neither the same antecedent nor the same consequent, it will be proper then to reduce them to the same antecedent, by setting off the ratio

to be subtracted from the antecedent of the other, thus : let it be required to subtract the ratio of  $C$  to  $D$  from the ratio of  $A$  to  $B$  : now the ratio of  $C$  to  $D$  set off from  $A$  reaches to  $\frac{AD}{C}$  ; therefore to subtract the ratio of  $C$  to  $D$  from the ratio of  $A$  to  $B$  is the same as to subtract the ratio of  $A$  to  $\frac{AD}{C}$  from the ratio of  $A$  to  $B$  ; but the ratio of  $A$  to  $\frac{AD}{C}$  subtracted from the ratio of  $A$  to  $B$ , a ratio of the same antecedent, leaves the ratio of  $\frac{AD}{C}$  to  $B$ , or of  $AD$  to  $BC$  ; therefore the ratio of  $C$  to  $D$  subtracted from the ratio of  $A$  to  $B$  leaves the ratio of  $AD$  to  $BC$ . The rule then is as follows :

*Whenever one ratio is to be subtracted from another, change the sign of the ratio to be subtracted by inverting its terms, and then the sum of this new ratio added to the other will be the same with the remainder of the intended subtraction.* Thus to subtract the ratio of  $C$  to  $D$  from the ratio of  $A$  to  $B$  is the same as to add the ratio of  $D$  to  $C$  to the ratio of  $A$  to  $B$  ; but the ratio of  $D$  to  $C$  added to the ratio of  $A$  to  $B$  gives the ratio of  $AD$  to  $BC$  by the last article ; therefore the ratio of  $C$  to  $D$  subtracted from the ratio of  $A$  to  $B$  leaves the ratio of  $AD$  to  $BC$ . For a further proof of this, we are to take notice, that in all subtraction whatever, the remainder and the quantity subtracted ought both together to make the quantity from whence the subtraction was made ; but in our case the remainder was the ratio of  $AD$  to  $BC$ , and the quantity subtracted was the ratio of  $C$  to  $D$ , and these two added together make the ratio of  $ACD$  to  $BCD$ , or of  $A$  to  $B$ , which is the ratio from whence the subtraction was made ; therefore the remainder in this case was rightly assigned.

For an example in numbers, let it be required to subtract the ratio of 4 to 5 from the ratio of 2 to 3 : now the ratio of 5 to 4 added to the ratio of 2 to 3 gives the ratio of 10 to 12, or of 5 to 6, by the last article ; therefore the ratio of 4 to 5 subtracted from the ratio of 2 to 3 leaves the ratio of 5 to 6, which may be confirmed thus : the ratio of 2 to 3 is the same with the ratio of 4 to 6, which contains the ratios of 4 to 5 and of 5 to 6 ; therefore, if the ratio of 4 to 5 be taken away, there will remain the other part, which is the ratio of 5 to 6.

Before I conclude this article, I ought to take notice that there is another way of conceiving the subtraction of ratios, which for its use in Physics and Mechanics ought not to be passed by in this place ; it is thus : the ratio of  $C$  to  $D$  added to the ratio of  $A$  to  $B$  constitutes the ratio of  $AC$  to  $BD$  ; therefore, *converso*, the ratio of  $C$  to  $D$  subtracted from the ratio of  $A$  to  $B$  must leave the ratio of  $\frac{A}{C}$  to  $\frac{B}{D}$ , because multiplication and division are as much the reverse of one another as addition and subtraction ; but this ratio of  $\frac{A}{C}$  to  $\frac{B}{D}$ , when reduced to integral terms, is the same with the ratio of  $AD$  to  $BC$  found before.

N B. *Wherever it is said that the ratio of  $A$  to  $B$  is compounded of the direct ratio of  $C$  to  $D$ , and of the inverse or reciprocal ratio of  $E$  to  $F$ , the meaning is, that the ratio of  $A$  to  $B$  is equal to the excess of the ratio of  $C$  to  $D$  above the ratio of  $E$  to  $F$ , or that  $A$  is to  $B$  as  $\frac{C}{E}$  to  $\frac{D}{F}$ , or as  $CF$  to  $DE$ .*

### *Of the Multiplication and Division of Ratios.*

297. If the ratio of  $A$  to  $B$  be added to itself, that is, to the ratio of  $A$  to  $B$ , the sum will be the ratio of  $A^2$  to  $B^2$  by the last article but one ; and this  
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being added again to the ratio of  $A$  to  $B$  gives the ratio of  $A^3$  to  $B^3$ , and so on; therefore the ratio of  $A^2$  to  $B^2$  is double, and the ratio of  $A^3$  to  $B^3$  triple, of the ratio of  $A$  to  $B$ . And universally, *The ratio of  $A^n$  to  $B^n$  is such a multiple of the ratio of  $A$  to  $B$  as is expressed by the number  $n$ .* Thus the ratio of  $A^4$  to  $B^4$  is four times the ratio of  $A$  to  $B$ , which I prove thus: the ratio of  $A$  to  $B$  reaches first from  $A^4$  to  $A^3B$ , 2dly, from  $A^3B$  to  $A^2B^2$ , 3dly, from  $A^2B^2$  to  $AB^3$ , and 4thly, from  $AB^3$  to  $B^4$ .

To give an example in numbers, I say that five times the ratio of 2 to 3 is the ratio of the fifth power of 2 to the fifth power of 3, that is, the ratio of 32 to 243: for the ratio of 2 to 3 reaches 1st from 32 to 48, 2dly, from 48 to 72, 3dly, from 72 to 108, 4thly, from 108 to 162, and 5thly, from 162 to 243. Thus much for multiplication.

*Division is the reverse of multiplication; and therefore as every ratio is doubled, or trebled, or quadrupled by squaring, or cubing, or square-squaring its terms, so every ratio is bisected, or trisected, or quadrisectioned by extracting the square, or cube, or square-square roots of its terms.* Thus half the ratio of 2 to 3 is the ratio of the square root of 2 to the square root of 3, that is (when reduced according to the first scholium in art. 179\*) the ratio of 40 to 49 nearly; which is further proved thus: the ratio of 40 to 49 is half the ratio of 1600 to 2401, by what was delivered in the former part of this article; but 1600 is to 2400 as 2 to 3; therefore 1600 is to 2401 as 2 to 3 very near.

But there is no necessity of a double extraction of the root in the division of a ratio, provided the ratio proposed be reduced to an equal one whose antecedent is unity. Thus 2 is to 3 as 1 to  $\frac{3}{2}$ , and therefore half the ratio of 2 to 3 is the ratio of 1 to  $\sqrt{\frac{3}{2}}$ , or the ratio of 1 to  $\sqrt{1.5}$ .

*From what has been said it appears that one ratio may be commensurate to another, and yet the terms of*

\* See the Quarto Edition, p. 283.

one incommensurate to the terms of the other : thus the ratio of 2 to 3 is certainly commensurate to the ratio of the square root of 2 to the square root of 3, the former being double of the latter ; and yet 2 and 3, the terms of the former ratio, are incommensurate to  $\sqrt{2}$  and  $\sqrt{3}$ , the terms of the latter.

Note. *Wherever it is said that A is to B in a sesquiplicate ratio of C to D, the meaning is, that the ratio of A to B is equal to  $\frac{3}{2}$  of the ratio of C to D :* therefore, in such a case, twice the ratio of A to B will be equal to three times the ratio of C to D ; but twice the ratio of A to B is the ratio of  $A^2$  to  $B^2$ , and three times the ratio of C to D is the ratio of  $C^3$  to  $D^3$  ; therefore if A be to B in a sesquiplicate ratio of C to D,  $A^2$  will be to  $B^2$  as  $C^3$  to  $D^3$ . Thus, in the revolutions of the primary planets about the Sun, and of the secondary planets about Jupiter and Saturn, their periodic times are said to be in a sesquiplicate ratio of their middle distances, that is, the squares of their periodic times are as the cubes of their middle distances.

*Another Way of Multiplying and Dividing small Ratios, that is, whose Terms are large in comparison of their difference.*

298. Before I deliver what I have to say upon this head, I shall only observe, that *If two intermediate quantities have always the same difference, the greater the quantities are, the nearer will their ratio approach towards a ratio of equality :* thus the difference betwixt 2 and 1 is the same with the difference betwixt 100 and 99 ; but the ratio of 2 to 1, or of 100 to 50 is much greater than the ratio of 100 to 99. By the help of this observation, and the following theorem, I shall endeavour to shew that small ratios may sometimes be doubled, or tripled, or bisected, or trisected, by more compendious ways than those that are taught in the last article ; and whenever they happen

to be so, they ought to be used, and frequently are used, rather than the other.

### A T H E O R E M.

*If there be two quantities whose difference is but small in comparison of the quantities themselves, and if so much be added to one and subtracted from the other as shall make their difference double, or triple, or half, or a third part of what it was before; I say then that the quantities after this alteration shall be in a duplicate, or a triplicate, or a subduplicate, or a subtriplicate ratio of that they were in before any such change was made, nearly.*

1<sup>st</sup>, Let there be two numbers 10 and 11, whose difference is 1; then if  $\frac{1}{2}$  be added to 11 and subtracted from 10, we shall have the numbers  $11\frac{1}{2}$  and  $9\frac{1}{2}$ , whose difference is 2: I say now that  $11\frac{1}{2}$  is to  $9\frac{1}{2}$  in a duplicate ratio of 11 to 10 nearly. For the ratio of  $11\frac{1}{2}$  to  $9\frac{1}{2}$  is resolvable into these two ratios, viz. the ratio of  $11\frac{1}{2}$  to  $10\frac{1}{2}$  and the ratio of  $10\frac{1}{2}$  to  $9\frac{1}{2}$ : now of these two ratios the former, to wit, that of  $11\frac{1}{2}$  to  $10\frac{1}{2}$ , is somewhat less than the ratio of 11 to 10, by the observation at the beginning of this article; and the latter, to wit, that of  $10\frac{1}{2}$  to  $9\frac{1}{2}$ , is somewhat greater than the ratio of 11 to 10, and the excess in this case is nearly equal to the defect in the former; therefore the sum of both these ratios put together, that is, the ratio of  $11\frac{1}{2}$  to  $9\frac{1}{2}$  will be very nearly equal to twice the ratio of 11 to 10.

2<sup>dly</sup>, As the difference betwixt 10 and 11 is 1, add 1 to 11 and subtract it from 10, and you will have the numbers 12 and 9, whose difference is 3: I say now that 12 will be to 9, or 4 to 3, in a triplicate ratio of 11 to 10 nearly. For the ratio of 12 to 9 is resolvable into these three ratios, to wit, the ratio of 12 to 11, the ratio of 11 to 10, and the ratio of 10 to 9; and of these three ratios, the first, to wit, that of 12 to 11, is somewhat less than the middle ratio of 11 to 10; and the last, to wit, that of 10 to 9, is about as much greater;

greater ; therefore the first and last ratios put together will make about twice the middle ratio of 11 to 10 ; therefore all these three ratios put together, to wit, the ratio of 12 to 9, will make three times the ratio of 11 to 10 nearly.

3dly, And if increasing the difference increases the ratio proportionably, then diminishing the difference ought to diminish the ratio proportionably, that is, if the difference be reduced to half, or a third part of what it was at first, the ratio ought to be so reduced : now as the difference between 10 and 11 is 1, add  $\frac{1}{4}$  to 10 and subtract it from 11, and you will have the numbers  $10\frac{1}{4}$  and  $10\frac{3}{4}$ , whose difference is  $\frac{1}{2}$ , and  $10\frac{1}{4}$  will be to  $10\frac{3}{4}$  in a subduplicate ratio of 10 to 11 nearly ; but if  $\frac{1}{3}$  be added to 10 and subtracted from 11, you will then have the numbers  $10\frac{1}{3}$  and  $10\frac{2}{3}$ , whose difference is  $\frac{1}{3}$  ; and  $10\frac{1}{3}$  will be to  $10\frac{2}{3}$  in a subtriplicate ratio of 10 to 11 nearly.

Let us now try how near the ratios here found approach to the truth. By the last article, the duplicate ratio of 10 to 11 is the ratio of 100 to 121, or of 1 to 1.2100 ; and according to the foregoing theorem it is the ratio of  $9\frac{1}{2}$  to  $11\frac{1}{2}$ , or of 19 to 23, or of 1 to 1.2105.

By the last article the triplicate ratio of 10 to 11 is the ratio of 1000 to 1331, or of 1 to 1.331 ; and according to the foregoing theorem it is the ratio of 9 to 12, or of 3 to 4, or of 1 to 1.333.

By the last article the subduplicate ratio of 10 to 11 is the ratio of 1 to the square root of  $\frac{11}{10}$ , or of 1 to 1.04881 ; and according to the foregoing theorem it is the ratio of  $10\frac{1}{4}$  to  $10\frac{3}{4}$ , or of 41 to 43, or of 1 to 1.04878.

By the last article the subtriplicate ratio of 10 to 11 is the ratio of 1 to the cube root of  $\frac{11}{10}$ , that is, of 1 to 1.03228 ; and according to the foregoing theorem it is the ratio of  $10\frac{1}{3}$  to  $10\frac{2}{3}$ , or of 31 to 32, or of 1 to 1.03226.

By

By these instances we see how near these ratios come up to the truth, even when the difference is no less than a tenth or an eleventh part of the whole : but if we suppose the difference to be the hundredth or the thousandth part of the whole, they will be much more accurate ; inasmuch that, to multiply or divide the ratio, it will be sufficient to encrease or diminish one of the numbers only. Thus 100 is to 102 in a duplicate, and to 103 in a triplicate, ratio of 100 to 101 ; and 100 is to  $100 + \frac{1}{2}$  in a subduplicate, and to  $100 + \frac{1}{3}$  in a subtriplicate, ratio of 100 to 101 nearly : and universally, *If  $A + z$  and  $A + y$  be any two quantities approaching infinitely near to the quantity A, the ratio of  $A + z$  to A will be to the ratio of  $A + y$  to A as the infinitely small difference  $z$  is to the infinitely small difference  $y$ .*

I shall draw only one example out of an infinite number that might be produced to shew the use of the foregoing proposition. Suppose then I have a clock that gains one minute every day ; how much must I lengthen the pendulum to set it right ? Let  $l$  be the present length of the pendulum, let  $x$  be the increment to be added to its length in order to correct its motion, and let  $n$  be the number of minutes in one day ; then it is plain that the pendulum  $l$  performs the same number of vibrations in the time  $n - 1$  that the pendulum  $l + x$  is to perform in the time  $n$ . Now Monsieur *Huygens* has demonstrated that the times, wherein different pendulums perform the same number of vibrations are in a subduplicate ratio of the lengths of those pendulums ; therefore  $n - 1$  must be to  $n$  in a subduplicate ratio of  $l$  to  $l + x$ , or (which comes to the same thing)  $l$  must be to  $l + x$  in a duplicate ratio of  $n - 1$  to  $n$  : but by the foregoing proposition, the duplicate ratio of  $n - 1$  to  $n$  is the ratio of  $n - \frac{3}{2}$  to  $n + \frac{1}{2}$ , or of  $2n - 3$  to  $2n + 1$  ; therefore  $l$  is to  $l + x$  as  $2n - 3$  is to  $2n + 1$ , that is, the pendulum must be lengthened in the proportion of  $2n - 3$  to  $2n + 1$  : but  $n$  the number of minutes in one day

is 1440; and therefore  $2n - 3$  is to  $2n + 1$  as 2877 is to 2881, or as 719 to 720 very near; therefore the pendulum must be lengthened in the proportion of 719 to 720. *Q. E. I.*

Had the duplicate ratio of  $n - 1$  to  $n$  been taken only by diminishing  $n - 1$  to  $n - 2$ , without meddling with the other number  $n$ , the conclusion would still have been the same, for then  $l$  would have been to  $l + x$  as  $n - 2$  to  $n$ , as 1438 to 1440, as 719 to 720.

Having now delivered what I intended concerning the composition and resolution of ratios, it remains that I say something further concerning the application of this doctrine, and then I shall make an end of the subject.

#### DEFINITION 4.

299. *If two variable quantities Q and R be of such a nature, that R cannot be increased or diminished in any proportion, but Q must necessarily be increased or diminished in the same proportion; as if R cannot be changed to any other value r, but Q must also be changed to some other value q, and so changed that Q shall always be to q in the same proportion as R to r; then is Q said to be as R directly, or simply as R.* Thus is the circumference of a circle said to be as the diameter; because the diameter cannot be increased or diminished in any proportion, but the circumference must necessarily be increased or diminished in the same proportion. Thus is the weight of a body said to be as the quantity of matter it contains, or proportionable to the quantity of matter; because the quantity of matter cannot be increased or diminished in any proportion, but the weight must be increased or diminished in the same proportion.

#### COROLLARY 1.

*If Q be as R directly, then è converso R must necessarily be as Q directly.* For let Q be changed to any

any other value  $q$ , and at the same time let  $R$  be changed to  $r$ ; then since  $Q$  is as  $R$ ,  $Q$  will be to  $q$  as  $R$  to  $r$ ; but if  $Q$  is to  $q$  as  $R$  is to  $r$ , then *vice versa*  $R$  will be to  $r$  as  $Q$  to  $q$ : since then  $Q$  cannot be changed to  $q$ , but  $R$  must be changed to  $r$ , and that in the same proportion, it follows by this definition that  $R$  is as  $Q$  directly.

## COROLLARY 2.

If  $Q$  be directly as  $R$ , and  $R$  be directly as  $S$ , then will  $Q$  be directly as  $S$ . For let  $S$  be changed to  $s$ , and at the same time  $R$  to  $r$ , and  $Q$  to  $q$ ; then since by the supposition  $R$  is as  $S$ ,  $R$  must be to  $r$  as  $S$  to  $s$ ; and since again  $Q$  is as  $R$ ,  $Q$  will be to  $q$  as  $R$  to  $r$ : since then  $Q$  is to  $q$  as  $R$  to  $r$ , and  $R$  is to  $r$  as  $S$  to  $s$ , it follows that  $Q$  will be to  $q$  as  $S$  to  $s$ , and consequently that  $Q$  will be as  $S$ .

## COROLLARY 3.

If  $Q$  be as  $R$ , and  $R$  be as  $S$ ; I say then that  $Q$  will be as  $R \pm S$ , and also as the square root of the product  $RS$ . For changing  $Q, R, S$ , into  $q, r, s$ , since  $R$  is as  $S$ , we shall have  $R$  to  $r$  as  $S$  to  $s$ ; whence by the twelfth and nineteenth of the fifth book of the Elements  $R$  will be to  $r$  as  $R \pm S$  is to  $r \pm s$ ; but  $Q$  is to  $q$  as  $R$  is to  $r$ , *ex hypothesi*; therefore  $Q$  is to  $q$  as  $R \pm S$  is to  $r \pm s$ ; therefore by this definition  $Q$  will be as  $R \pm S$ . Again, since  $R$  is as  $S$ ,  $R^2$  will be as  $RS$ , and  $R$  as  $\sqrt{RS}$ ; but  $Q$  is as  $R$ ; therefore by the last corollary  $Q$  will be as  $\sqrt{RS}$ .

## COROLLARY 4.

If any variable quantity as  $Q$  be multiplied by any given number as 5; I say then that  $5Q$  will be as  $Q$ . For it will be impossible for  $Q$  to be increased or diminished in any proportion, but  $5Q$  must be increased or diminished in the same proportion: if  $Q$  in any one case be double of  $Q$  in another, then  $5Q$  in the

the former case must be double of  $5Q$  in the latter, and so on; therefore  $5Q$  is as  $Q$ .

## COROLLARY 5.

If  $Q$  be as  $R$ , then  $Q^2$  will be as  $R^2$ ,  $Q^3$  as  $R^3$ ,  $\sqrt{Q}$  as  $\sqrt{R}$ , &c. For let  $R^2$  be changed in the proportion of  $D$  to  $E$ ; then will  $R$  be changed in the proportion of  $\sqrt{D}$  to  $\sqrt{E}$ ; but  $Q$  is as  $R$ ; therefore  $Q$  will also be changed in the proportion of  $\sqrt{D}$  to  $\sqrt{E}$ ; therefore  $Q^2$  will be changed in the proportion of  $D$  to  $E$ : since then  $R^2$  cannot be changed in any proportion, suppose of  $D$  to  $E$ , but  $Q^2$  must necessarily be changed in the same proportion, it follows from this definition that  $Q^2$  is as  $R^2$ : and the reasoning is the same in all other cases.

## COROLLARY 6.

If  $Q$ ,  $R$ , and  $S$ , be three variable quantities, and  $Q$  be as the product or rectangle  $RS$ ; I say then, that  $\frac{Q}{R}$  will always be as  $S$ , and  $\frac{Q}{S}$  as  $R$ , and that  $\frac{Q}{RS}$  will be a given quantity, or (which is chiefly meant by that phrase) that the quantity  $\frac{Q}{RS}$  will always be the same, be the values of  $Q$ ,  $R$ , and  $S$ , what they will. For since  $Q$  is as  $RS$ ,  $Q$  cannot be increased or diminished in any proportion, but  $RS$  must be increased or diminished in the same proportion; therefore  $\frac{Q}{R}$  cannot be increased or diminished in any proportion, but  $\frac{RS}{R}$  or  $S$  must be increased or diminished in the same proportion; therefore  $S$  is as  $\frac{Q}{R}$ , and  $\frac{Q}{R}$  as  $S$ : and by a like proof,  $R$  will be as  $\frac{Q}{S}$ , and  $\frac{Q}{S}$  will be as  $R$ : but if  $\frac{Q}{S}$  be as  $R$ , then

then dividing both sides by  $R$ , we shall have  $\frac{Q}{RS}$  as 1; but 1 is a quantity that neither increases nor diminishes, but is always the same; therefore the quantity  $\frac{Q}{RS}$  will always be the same; and for the same reason, If  $Q$  be as any single quantity, suppose  $R$ ,  $\frac{Q}{R}$  will always be the same, let  $Q$  and  $R$  be what they will.

#### COROLLARY 7.

If there be four variable quantities  $A, B, C, D$ , all in numbers, whereof  $A$  is as  $B$ , and  $C$  is as  $D$ ; I say then that the product  $AC$  will be as the product  $BD$ . For since  $A$  is as  $B$ ,  $AC$  will be as  $BC$ , and since  $C$  is as  $D$ ,  $BC$  will be as  $BD$ ; therefore by the second corollary  $AC$  will be as  $BD$ ; that is,  $AC$  in one case will be to  $AC$  in any other as  $BD$  in the former case is to  $BD$  in the latter.

#### DEFINITION 5.

300. If two variable quantities  $Q$  and  $R$  be of such a nature, that  $R$  cannot be increased in any proportion whatever, but  $Q$  must necessarily be diminished in a contrary proportion, or that  $R$  cannot be diminished in any proportion whatever, but  $Q$  must necessarily be increased in a contrary proportion; in a word, if  $R$  cannot be changed in a proportion of  $D$  to  $E$ , but  $Q$  must necessarily be changed in the proportion of  $E$  to  $D$ ; then is  $Q$  said to be as  $R$  inversely or reciprocally. Thus if a spherical body be viewed at any considerable distance, the apparent diameter is said to be reciprocally as the distance, because the greater the distance is, the less will be the apparent diameter, and vice versa. Thus if a globe be supposed to move uniformly about its axis, the periodical time of this motion is said to be reciprocally as the velocity with which the globe circulates (for the quicker the circulation

ulation is, the sooner it will be over); which is as much as to say, that the greater the velocity is with which the globe circulates, the less will be the periodical time of one revolution, and *vice versa*. Thus if the numerator of a fraction continues always the same whilst the denominator is supposed to vary, that fraction is said to be reciprocally as its denominator, because the greater the denominator is, the less will be the value of the fraction, and *vice versa*.

#### COROLLARY I.

If  $Q$  be reciprocally as  $R$ , then è converso  $R$  will be reciprocally as  $Q$ . For let  $Q$  be changed in the proportion of  $D$  to  $E$ , and at the same time let  $R$  be changed in the proportion of  $A$  to  $B$ ; then since  $Q$  is reciprocally as  $R$ ,  $Q$  must be changed in the proportion of  $B$  to  $A$ ; but  $Q$  was changed in the proportion of  $D$  to  $E$ ; therefore  $B$  must be to  $A$  as  $D$  to  $E$ ; therefore, inversely,  $A$  must be to  $B$  as  $E$  to  $D$ ; but  $R$  was changed in the proportion of  $A$  to  $B$  by the supposition; therefore  $R$  was changed in the proportion of  $E$  to  $D$ . Since then  $Q$  cannot be changed in any proportion, suppose of  $D$  to  $E$ , but  $R$  must necessarily be changed in the contrary proportion of  $E$  to  $D$ , it follows from this definition that  $R$  must be reciprocally as  $Q$ .

#### COROLLARY 2.

If  $Q$  be directly as  $R$ , and  $R$  be reciprocally as  $S$ , then  $Q$  must be reciprocally as  $S$ . For let  $S$  be changed in the proportion of  $D$  to  $E$ ; then since  $R$  is reciprocally as  $S$ ,  $R$  must be changed in the proportion of  $E$  to  $D$ ; but  $Q$  is directly as  $R$  by the supposition; therefore  $Q$  must also be changed in the proportion of  $E$  to  $D$ . Since then  $S$  cannot be changed in the proportion of  $D$  to  $E$ , but  $Q$  must necessarily be changed in the proportion of  $E$  to  $D$ , it follows from this definition that  $Q$  is reciprocally as  $S$ .

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## COROLLARY 3.

By a like way of reasoning, if  $Q$  be reciprocally as  $R$ , and  $R$  be reciprocally as  $S$ ,  $Q$  will be directly as  $S$ .

## COROLLARY 4.

If two variable quantities  $Q$  and  $R$  be of such a nature that their product or rectangle  $QR$  is always the same; I say then that  $Q$  will be reciprocally as  $R$ . For since  $QR$  is always the same, it will be as the number 1, which neither increases nor diminishes; but if  $QR$  be as one, then  $Q$  will be as the fraction  $\frac{1}{R}$  by the sixth corollary to the fourth definition. Since then  $Q$  is directly as the fraction  $\frac{1}{R}$ , and the fraction  $\frac{1}{R}$  is reciprocally as its denominator  $R$  by this definition, it follows from the second corollary that  $Q$  will be reciprocally as  $R$ .

## COROLLARY 5.

Every fraction is reciprocally as the same fraction inverted. Thus the fraction  $\frac{R}{S}$  is reciprocally as the fraction  $\frac{S}{R}$ . This is evident from the last corollary; for if the fractions  $\frac{R}{S}$  and  $\frac{S}{R}$  be multiplied together, their product will always be unity, let  $R$  and  $S$  be what they will.

## COROLLARY 6.

If  $Q$  be reciprocally as  $R$ , or reciprocally as  $\frac{R}{Y}$ , then

then  $Q$  will be directly as  $\frac{1}{R}$ . For since  $Q$  is reciprocally as  $\frac{R}{1}$ , and  $\frac{R}{1}$  is reciprocally as  $\frac{1}{R}$  by the last corollary, it follows from the third corollary that  $Q$  will be directly as  $\frac{1}{R}$ . For the same reason, If  $Q$  be reciprocally as  $\frac{1}{R}$ , it will be directly as  $R$ .

### DEFINITION 6.

301. If any quantity as  $Q$  depends upon several others as  $R, S, T, V, X$ , all independent of one another, so that any one of them may be changed singly without affecting the rest; and if none of the quantities  $R, S, T$ , can be changed singly, but  $Q$  must be changed in the same proportion, nor any of the quantities  $V, X$ , but  $Q$  must be changed in a contrary proportion; then is  $Q$  said to be as  $R$ , and  $S$ , and  $T$ , directly, and as  $V$  and  $X$  reciprocally or inversely. Thus the fraction  $\frac{RST}{VX}$  is said to be as  $R$ , and  $S$ , and  $T$ , directly, and as  $V$  and  $X$  inversely, because none of the factors belonging to the numerator can be changed, but the value of the fraction must be changed in the same proportion, and none of the factors belonging to the denominator can be changed, but the value of the fraction must be changed in a contrary proportion.

N. B. If  $Q$  be as  $R$ , and  $S$ , and  $T$ , directly, without any reciprocals, then it is said to be as  $R$ , and  $S$ , and  $T$ , conjunctim, jointly.

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302. If  $Q$  be as  $R$ , and  $S$ , and  $T$ , directly, and as  $V$  and  $X$  reciprocally; and if the quantities  $R, S, T, V, X$ , be changed into  $r, s, t, v, x$ , and so  $Q$  into  $q$ ; I say then that the ratio of  $Q$  to  $q$  will be equal to the excess

excess of all the direct ratios taken together above all the reciprocal ones taken together: as if the ratios of  $R$  to  $r$ , of  $S$  to  $s$ , and of  $T$  to  $t$  (which I call direct ratios), when added together make the ratio of  $A$  to  $B$ ; and if the ratios of  $V$  to  $v$ , and of  $X$  to  $x$  (which I call reciprocal ratios), when added together make the ratio of  $C$  to  $D$ ; I say then that the ratio of  $Q$  to  $q$  will be equal to the excess of the ratio of  $A$  to  $B$  above the ratio of  $C$  to  $D$ .

For supposing all but  $R$  to continue the same, let  $R$  be changed into  $r$ ; then will  $Q$  be changed from its first value in the ratio of  $R$  to  $r$  by the hypothesis: let now  $r$ ,  $T$ ,  $V$ ,  $X$ , continue, and let  $S$  be changed into  $s$ : then will  $Q$  be changed from its last value in the ratio of  $S$  to  $s$ : in like manner if  $T$  be changed into  $t$ , *cæteris paribus*,  $Q$  will be changed from its last value in the ratio of  $T$  to  $t$ : therefore if  $R$ ,  $S$ ,  $T$ , be changed into  $r$ ,  $s$ ,  $t$ ,  $Q$  will be changed from one value to another in a ratio compounded of all the direct ratios of  $R$  to  $r$ , of  $S$  to  $s$ , and of  $T$  to  $t$ ; that is,  $Q$  will be changed in the ratio of  $A$  to  $B$ . This being so, let us now imagine  $V$  to be changed, *cæteris paribus*, into  $v$ ; then will  $Q$  be further changed in the ratio of  $v$  to  $V$ ; and if after this we imagine  $X$  to be changed into  $x$ ,  $Q$  will be changed in the proportion of  $x$  to  $X$ , and will now be arrived at its last value  $q$ : therefore if to the ratio of  $A$  to  $B$  you add the ratios of  $v$  to  $V$  and of  $x$  to  $X$ , you will have the ratio of  $Q$  to  $q$ : but to add the ratio of  $v$  to  $V$  is the same thing as to subtract the ratio of  $V$  to  $v$  by art. 296; and so again, to add the ratio of  $x$  to  $X$  is the same as to subtract the ratio of  $X$  to  $x$ ; therefore if from the ratio of  $A$  to  $B$  you subtract the ratios of  $V$  to  $v$  and of  $X$  to  $x$ , you will have the ratio of  $Q$  to  $q$ ; but the ratios of  $V$  to  $v$ , and of  $X$  to  $x$ , when added together, make the ratio of  $C$  to  $D$ , *ex hypothesis*; therefore if from the ratio of  $A$  to  $B$  you subtract the ratio of  $C$  to  $D$ , there will remain the ratio of  $Q$  to  $q$ ; therefore the ratio of

$\mathcal{Q}$  to  $q$  is the excess of the ratio of  $A$  to  $B$  above the ratio of  $C$  to  $D$ ; or (which is the same thing)  $\mathcal{Q}$  is to  $q$  in a ratio compounded of the ratio of  $A$  to  $B$  directly, and of the ratio of  $C$  to  $D$  inversely. See art. 296.

This is upon a supposition that the quantities  $R, S, T, V, X$ , were changed into  $r, s, t, v, x$ , one after another in time: but since the ratio of  $\mathcal{Q}$  to  $q$  does not depend upon the intervals of time between the several changes, but will be the same whether those intervals be greater or less, it follows that the ratio of  $\mathcal{Q}$  to  $q$  will be the same as if all these changes had been made at once.  $\mathcal{Q}. E. D.$

### COROLLARY I.

If the quantities  $R, S, T, V, X$ , and consequently  $A, B, C, D$ , be expressed by numbers, as they must be before they can be of use in any computation; then the ratio of  $A$  to  $B$  will be the ratio of  $RST$  to  $rst$ , and the ratio of  $C$  to  $D$  will be the ratio of  $VX$  to  $vx$ ; and the excess of the ratio of  $A$  to  $B$  above the ratio of  $C$  to  $D$  will be the ratio of  $\frac{RST}{VX}$  to  $\frac{rst}{vx}$ ; (see the second way of subtracting ratios in art. 296;) therefore, in this case,  $\mathcal{Q}$  will be to  $q$  as the fraction  $\frac{RST}{VX}$  is to the fraction  $\frac{rst}{vx}$ . Since then the fraction  $\frac{RST}{VX}$  cannot be changed into  $\frac{rst}{vx}$  but at the same time  $\mathcal{Q}$  must be changed into  $q$ , and so changed that  $\mathcal{Q}$  will be to  $q$  as  $\frac{RST}{VX}$  is to  $\frac{rst}{vx}$ , it follows from the fourth definition that  $\mathcal{Q}$  will be as the fraction  $\frac{RST}{VX}$ ; and consequently that  $\mathcal{Q}$  in any one case will be to  $\mathcal{Q}$  in any other as the fraction  $\frac{RST}{VX}$  in

in the former case is to the fraction  $\frac{RST}{VX}$  in the latter.

## COROLLARY 2.

*If there be no reciprocals, then Q will be as the product of all the direct terms, that is, as the product RS if there be two of them, or as the product RST if there be three of them, &c.*

## SCHOLIUM.

In the demonstration of the foregoing proposition, as well as in the sixth definition, it was supposed, that the quantities  $R, S, T, V, X$ , upon which  $Q$  depended, were themselves entirely independent of one another, so as that any of them might be changed singly without affecting the rest; and in such a case, if  $Q$  be as  $R$  and  $S$  directly, it may be concluded to be as the product  $RS$ . But this conclusion must not be carried farther than can be justified by the demonstration: for if in any case the quantities  $R$  and  $S$  should not be independent, if neither of them can be changed whilst the other continues the same, then though no change can be made either in  $R$  or  $S$  but what will make a proportionable change in  $Q$ , yet here  $Q$  must not be said to be as the product  $RS$ . As for example, let  $Q$  be an arc of a circle subtending at the distance  $R$  an angle whose quantity is represented by  $S$ ; then it is plain that neither  $R$  nor  $S$  can be changed singly, but  $Q$  must be changed proportionably; it is plain also that either  $R$  or  $S$  may be changed singly whilst the other remains the same; and therefore in this case it is lawful to conclude that  $Q$  is as the product  $RS$ . But let us now suppose  $Q$  to be the circumference of a circle whose radius is  $R$ , and let  $S$  be the side of a regular polygon of any given sort inscribed in that circle; as for instance, let  $S$  be the side of an inscribed square: here then it is plain that neither  $R$  nor  $S$  can be changed, but  $Q$  must be

changed proportionably; and yet if we should conclude in this case that  $\mathcal{Q}$  is as  $RS$ , the illation would be false, because  $R$  and  $S$  have here as much dependence upon one another as  $\mathcal{Q}$  upon both; for every one knows that the *radius* of a circle cannot be increased or diminished in any proportion, but the side of a square inscribed in that circle must be increased or diminished in the same proportion: in this case it may be concluded that  $\mathcal{Q}$  is as  $R + S$ , or as  $R - S$ , or as the square root of  $RS$  by the third corollary in art. 299; but it must by no means be allowed that  $\mathcal{Q}$  is as  $RS$ ; for should  $\mathcal{Q}$  be as  $RS$ , since in this case  $S$  is as  $R$ , and consequently  $RS$  as  $R^2$ ,  $\mathcal{Q}$  would be as  $R^2$  by the second corollary in art. 299, which contradicts the supposition that  $\mathcal{Q}$  is as  $R$ .

*Examples to illustrate the foregoing Theorem, where direct Ratios are only concerned.*

303. Ex. 1. *If a body moves for any time with any uniform velocity through any space, that space will be as the time and velocity jointly.* For if we suppose the velocity to be the same in all cases, but the time to differ, then the space described will be greater or less in proportion as the time is so, and therefore will be as the time: on the other hand, if we suppose the time to be the same in all cases, and the velocity to differ, then the space described in these equal times will be greater or less as the velocity is so, and consequently will be as the velocity: lastly, let us suppose both the time and velocity to vary; then the space will vary upon both these accounts, and therefore will vary in a ratio equal to the ratio wherein the time varies, and the ratio wherein the velocity varies put together; that is, the space in any one case will be to the space in any other in a ratio compounded of the ratio of the time in the former case to the time in the latter, and of the velocity in the former case to the velocity in the latter. This is universal; but if we suppose

suppose the time and velocity to be expressed by numbers, we must then say that the space described is as the product of the number representing the time multiplied into the number representing the velocity, by the second corollary in the last article; or that the space described in any one case is to the space described in any other as the product of the time and velocity in the former case is to a like product in the latter.

Ex. 2. *The quantity of matter in any body depends upon two things, viz. its magnitude and density (where by density I mean the compactness or closeness of its matter.)* For if two bodies of equal densities but of unequal magnitudes be compared, one body must have more matter than the other, or less, according as its solid content is greater or less, that is, according as its magnitude is greater or less; therefore in this case the quantities of matter in any two bodies thus compared will be as their magnitudes: on the other hand, if two bodies of the same magnitude but of different densities be compared, their quantities of matter will be as their densities, because the closer the parts of a body are, so much more matter will be crowded into the same space; therefore, if the bodies be different both in magnitude and density, the quantity of matter in one body will be to the quantity of matter in the other in a ratio compounded of the ratio of the magnitude of one body to the magnitude of the other, and of the ratio of the density of the former body to the density of the latter; and therefore, if these quantities be represented by numbers, the quantity of matter in any body will be as its magnitude and density multiplied together. Thus if  $D$  and  $d$  be the diameters of two globes whose densities are as  $E$  to  $e$ , the quantity of matter in the former globe will be to the quantity of matter in the latter as  $D^3 \times E$  is to  $d^3 \times e$ ; for the solid contents of all globes are as the cubes of their diameters.

Ex. 3. *The momentum, or force, or impetus with which a body moves, and with which it will strike any*

obstacle that lies in its way to oppose or stop it, is as the velocity of the motion and the quantity of matter in the body jointly. For the same quantity of matter moving with different velocities will strike an obstacle with forces proportionable to the velocities: on the other hand, different quantities of matter moving with the same velocity will strike with forces proportionable to their matter; a double body will strike with a double force, &c.; therefore, in the case where the velocity is the same, the *momentum* of a body is as the quantity of matter it contains; and in the case where the quantity of matter is the same, the *momentum* is as the velocity; therefore, if neither the velocity nor the matter be the same, the *momentum* will be as the matter and velocity jointly; and, in numbers, as the product of the number expressing the matter multiplied into the number expressing the velocity.

Ex. 4. *If a heavy body be suspended perpendicularly upon a lever (by which I mean an inflexible rod moving about a fixed point in the middle), the momentum or efficacy of that body to turn the lever about its center is, cæteris paribus, as the weight of the body and as the distance of the point of suspension from the center of the lever jointly.* For if we suppose this distance to be the same, the *momentum* of the body to turn the lever must be greater or less according as its weight is so, from whence that *momentum* arises: on the other hand, if we suppose the weight to be always the same, but to be removed, sometimes farther from, and sometimes nearer to, the center, the *momentum* of the body to turn the lever will be greater or less in proportion to the distance of the point of suspension from the center of the lever, as is demonstrated in Mechanics, and may easily be tried by experience: therefore universally, the *momentum* of the body will be as this distance and the weight of the body jointly; and in numbers is as the product of the weight multiplied into the distance.

To

To illustrate this, I shall put the following question. Let a body weighing five pounds be suspended at the distance of six inches from the center of a lever, and let another body of seven pounds be suspended on the same side of the center at the distance of eight inches; then let a third body of nine pound weight be suspended on the other side of the center at the distance of ten inches: *Quere*, whether will these bodies sustain each other in *æquilibrio* or not; and if not, on which side will the lever dip, and with what *momentum*?

To resolve this, since we are at liberty to represent any one of these *momenta* by what numbers we please, provided the rest be represented proportionably, let us represent the *momentum* of the nine pound body by the product of its weight and distance multiplied together, that is, by  $9 \times 10$  or 90; then must the other *momenta* be represented by like products, or they would not be represented by numbers proportionable to them: therefore the *momentum* of the five pound body will be  $5 \times 6$  or 30, and that of the seven pound body  $7 \times 8$  or 56; and therefore the sum of the *momenta* on this side the center acting the same way will be 86: whence now it plainly appears that the lever will dip on the side of the nine pound body, because 90, the *momentum* on that side, is greater than 86, the sum of the *momenta* on the other side: and since the excess of 90 above 86 is 4, it follows that 4 will be the difference of the *momenta* on one side and the other; insomuch that if any one sustains this lever immoveable, he will sustain the same force as if all the weights now upon the lever were taken away, and a single pound weight was suspended at the distance of four inches from the center of the lever: therefore when all the weights were upon the lever, if a single pound weight had been suspended at four inches distance, and on the same side of the center with the other two bodies whose weights were five and seven pounds, the whole system would then have consisted in *æquilibrio*.

Upon this theorem, that the force of a body upon a lever is as its weight and distance from the center multiplied together, is founded the method of finding the centers of gravity of bodies, or the center of gravity of any system of bodies; let their places or positions be what they will: but I must not carry this matter any farther.

Ex. 5. *If a globe be made to move uniformly in an uniform fluid, the resistance it will meet with in any given time by impinging against the particles of the fluid, will be as the density of the fluid, and as the square of the diameter of the globe, and as the square of the velocity it moves with jointly.*

To determine rightly in this case, we must here do what we all along have done, and what we always must do in like cases; that is, we must take the whole to pieces, examine every particular circumstance by itself, *cæteris paribus*, and then put them all together. First then let us suppose the same globe to move with the same velocity, but sometimes in a denser fluid, and sometimes in a rarer; then it is plain that the denser the fluid is, the more particles of it the body will be likely to meet with in any given time, and consequently the greater resistance it will suffer from them; therefore the resistance of the body, *cæteris paribus*, will be as the density of the fluid. In the next place let us suppose different globes to move in the same fluid, and with the same velocity; then, since the resistance of these globes arises only from the surfaces, or rather from half their surfaces, and since the surfaces of all globes are as the squares of their diameters, it follows that the resistance these globes meet with will be as the squares of their diameters. Lastly, let us suppose the same globe to move in the same fluid with different velocities; then it is plain that a globe which moves with a double velocity will strike twice as many particles of the fluid in any given time, as it would if it was to move with a single velocity: but

but if the body strikes twice as many particles, then twice as many particles will strike it, whence arises the resistance; therefore the resistance of a body moving with a double velocity is upon this account double of what it would have been in the case of a single velocity: but this is not all; for it will not only strike twice as many particles, but it will strike every particle with twice the force in this case of what it would in the case of a single velocity; and therefore, since action and reaction are always equal, and since it is the reaction of the medium that creates the resistance, it follows that a body moving with a double velocity meets with four times the resistance of what it would meet with when moving with a single velocity. In like manner, a body that moves with a triple velocity will act three times as strong upon three times the number of particles, and therefore will suffer nine times the resistance of what it would suffer with a single velocity; therefore the same globe moving in the same medium with different velocities will meet with a resistance proportionable to the square of the velocity it moves with. Put now all these considerations together, and the resistance of a globe moving uniformly in an uniform fluid (I mean that resistance which arises from the globe's impinging against the particles of the medium) will be as the density of the medium, as the square of the diameter of the globe, and as the square of the velocity it moves with jointly. Thus if two globes, whose diameters are  $D$  and  $d$ , move with velocities which are to one another as  $V$  to  $v$  in two fluids whose densities are as  $E$  to  $e$ , the resistance of the former will be to the resistance of the latter as  $V^2 \times D^2 \times E$  is to  $v^2 \times d^2 \times e$ .

*Other Examples, wherein direct and reciprocal Ratios are mixed together.*

304. Ex. 6. *If a body be put into motion by any force directly applied, whether this force be a single impulse acting at once, or whether it be divided into several impulses acting successively; I say that the last velocity of this motion will be as the moving force directly, and as the quantity of matter in motion reciprocally.* For if different forces be applied to the same quantity of matter, the greater the force is, the greater will be the velocity, and *vice versa*; therefore in this case the velocity will be as the *vis motrix*: but if we suppose the same force to be applied to different quantities of matter, then the greater the quantity of matter is, the less will be the velocity, and *vice versa*, which I thus demonstrate. Suppose the moving force  $M$ , when applied to a certain quantity of matter as  $Q$ , will produce the velocity  $V$ ; I say then that the same force  $M$ , applied to a quantity of matter equal to  $2Q$ , will only produce a velocity equal to  $\frac{1}{2}V$ : for  $M$  acting upon  $2Q$  will produce the same velocity as  $\frac{1}{2}M$  acting upon  $1Q$ ; but  $\frac{1}{2}M$  acting upon  $Q$  will produce a velocity equal to  $\frac{1}{2}V$ , because by the supposition  $M$  acting upon  $Q$  will produce the velocity  $V$ ; therefore  $M$  acting upon  $2Q$  will produce a velocity equal to  $\frac{1}{2}V$ ; and for the same reason,  $M$  acting upon  $3Q$  will produce a velocity equal to  $\frac{1}{3}V$ , &c.; therefore, if the *vis motrix* be the same, the velocity of the motion produced will be reciprocally as the quantity of matter: therefore universally, the velocity will be as the *vis motrix* directly, and as the quantity of matter inversely. As if  $M$  be changed into  $m$ ,  $Q$  into  $q$ , and so  $V$  into  $v$ , the ratio of  $V$  to  $v$  will be equal to the excess of the ratio of  $M$  to  $m$  above the ratio of  $Q$  to  $q$ . In numbers thus;  $V$  will be to

$v$  as  $\frac{M}{Q}$  is to  $\frac{m}{q}$ ; see the first corollary in art. 302.

Otherwise thus: the *momentum* or *impetus* with which a body moves, is the force with which it will strike an object that lies in its way to stop it; therefore since action and reaction are equal, the force necessary to destroy any motion must be equal to the *momentum* with which the body moves: but the force necessary to destroy any motion is equal to the force that produced it, which we call the *vis motrix*; therefore in all motion whatever, the *vis motrix* must be equal to the *momentum*, and must be as the quantity of matter in the body moved multiplied into the velocity of the motion, because the *momentum* is so; see the last article, example the 3d: therefore  $M$  will always be as  $V \times Q$  and  $V$  as  $\frac{M}{Q}$ .

If  $M$  be as  $Q$ , then  $\frac{M}{Q}$  will be a standing quantity, and therefore the velocity  $V$  in this case will always be the same. Thus if the weights of all bodies be proportionable to the quantities of matter they contain, they will be equally accelerated in equal times; and *vice versa*, if all bodies, how different soever in the kinds and quantities of matter, be equally accelerated in equal times (as by undoubted experiments upon pendulums we find they are, setting aside the resistance of the air), it follows that the weights of bodies are proportionable to their quantities of matter only, without depending upon their forms, constitutions, or any thing else.

Ex. 7. *The velocity of a planet moving uniformly in a circle round the Sun is as its distance from the center of the Sun directly, and as its periodical time inversely.* For if two planets at different distances from the Sun perform their revolutions in the same time, that planet must move with the greatest velocity that has the greatest circumference to describe; therefore  
in

in this case, where the periodical time is given or always the same, the velocity of the planet must be as the circumference of the circle to be described: but the circumference of every circle is as its diameter or semidiameter; therefore, if the periodical time be given, the velocity of a planet must be as its distance from the Sun directly. Let us now suppose two planets revolving at the same distance from the Sun, but in different periodical times; then it is plain that the swifter planet will perform its revolution in less time, and *vice versa*; and therefore, if the distance be given, the velocity will be reciprocally as the periodical time. Put both these cases together, and the velocity of a planet moving uniformly round the Sun will be as its distance from the center of the Sun directly, and as its periodical time inversely. Thus the Earth's distance from the Sun is to that of Jupiter as 10 to 52 nearly; and the Earth's periodical time is to that of Jupiter as 1 year to 12 years nearly, or as 1 to 12; therefore the Earth's velocity is to Jupiter's velocity as  $\frac{10}{1}$  is to  $\frac{52}{12}$ , or as 120 to 52, or as 30 to 13.

This way of reasoning is applicable to all bodies moving uniformly in circles, let the law of their motions be what it will. But if (as that accurate Astronomer *Kepler* has demonstrated) the planetary motions be so tempered that their periodical times are in a sesquiplicate ratio of their distances, or (which is the same thing by art. 297) that the squares of their periodical times are as the cubes of their distances, we shall then have a more simple way of expressing the velocity of a planet thus: let  $V$  be the velocity, and  $D$  the distance of any planet from the Sun, and let  $T$  be the periodical time; then since, from what has been said,  $V$  is as  $\frac{D}{T}$ , we shall have  $V^2$  as  $\frac{D^2}{T^2}$ ; but, according to *Kepler's*

ler's proposition,  $T^2$  is as  $D^3$ , and  $\frac{D^2}{T^2}$ , as  $\frac{D^2}{D^3}$ , or as  $\frac{1}{D}$ ; therefore  $V^2$  is as  $\frac{1}{D}$ , and  $V$  as  $\frac{1}{\sqrt{D}}$ ; that is, in this case, the velocity of a planet is reciprocally in a subduplicate ratio of its distance from the Sun. So the velocity of a planet, whose distance is  $D$ , is to the velocity of a planet, whose distance is  $d$ , as  $\sqrt{d}$  is to  $\sqrt{D}$ , or as 1 is to  $\sqrt{\frac{D}{d}}$ .

Ex. 8. *If a wheel turns uniformly about its axis, the time of one round will be as the diameter of the wheel directly, and as the absolute velocity of every point in the circumference of the wheel inversely.* For if the circumference of a great wheel moves with the same velocity as the circumference of a small one, the periodical time of the former wheel will be as much greater in proportion than the periodical time of the latter as the circumference of the former wheel is greater than the circumference of the latter, or as the diameter of the former is greater than the diameter of the latter; therefore if the velocity of the wheel's circumference be given, the periodical time will be as the diameter of the wheel directly: let us now suppose the velocity of the circumference of the same wheel to be in any case increased; then will the periodical time be diminished in a contrary proportion, and *vice versa*; therefore if the diameter of a wheel be given, the periodical time will be reciprocally as the velocity of the circumference; therefore if neither the diameter nor the velocity of the circumference be given, the periodical time will be as the diameter of the wheel directly, and as the absolute velocity of every point in the circumference inversely. In numbers the periodical time will be as  $\frac{D}{V}$ .

Ex. 9. *The relative gravity of any species of bodies is as the absolute weight of any body of that species directly, and as its magnitude inversely*; where by the magnitude or bulk of a body is meant the quantity of space it takes up, and not the quantity of matter it contains.

All bodies of the same kind are supposed to weigh in proportion to their magnitudes; and therefore if a body of any one kind be compared with a body of the same magnitude of another kind, the proportion of their weights will always be the same, let their common magnitude be what it will; and hence arises the comparison in general of the weight of one species of bodies with the weight of another: if a cubic inch of gold be 19 times as heavy as a cubic inch of water, then a cubic foot of gold will be 19 times as heavy as a cubic foot of water, &c.; and so we pronounce in general that gold is 19 times as heavy as water, though we mean bulk for bulk. In this sense therefore may any one species of bodies be said to be heavier or lighter than another, in proportion as any one body of the former species is heavier or lighter than a body of the same magnitude of the latter, which is the same in effect with the first part of my assertion. Let us now compare bodies of the same weight, but of different magnitudes; and then it will appear that the specific gravities of these bodies, that is, of the several species to which they belong, will be reciprocally as the magnitudes of the bodies compared: thus if a cubic inch of gold be as heavy as 19 cubic inches of water, then the specific gravity of gold will be to the specific gravity of water, not as 1 to 19, but as 19 to 1; for if one cubic inch of gold be as heavy as 19 cubic inches of water, then 1 cubic inch of gold will be 19 times as heavy as 1 cubic inch of water; and therefore, from what has been said in the former case, the specific gravity of gold will be to the specific gravity of water as 19 to 1. Put both these cases together, and the relative gravity of any species of bodies will be as the absolute weight of any one body

body of that species directly, and as its magnitude inversely. Thus if in numbers  $P$  and  $p$  be the weights of two globes whose diameters are  $D$  and  $d$ , the specific gravities of the metals out of which these two globes were formed are as  $\frac{P}{D^3}$  to  $\frac{p}{d^3}$ .

Ex. 10. If a body as  $A$  gravitates toward the center of a planet as  $B$  at the distance  $D$ ; I say then that the weight of  $A$  will be as the quantity of matter in  $A$  directly, and as the quantity of matter in  $B$  directly, and as the square of the distance  $D$  inversely. For the weight of the whole body  $A$  towards  $B$  arises, *cæteris paribus*, from the weight of all its parts; and therefore in such a case will be as the quantity of matter in  $A$ . Again, the weight of  $A$  towards the whole planet  $B$  arises, *cæteris paribus*, from the weight of  $A$  to all the parts of  $B$ ; and therefore in such a case will be as the quantity of matter in  $B$ . Lastly, if the quantities of matter in  $A$  and  $B$  continue the same, and the distance  $D$  be supposed to vary, the great *Newton* has demonstrated that the weight of  $A$  towards  $B$  will be reciprocally as the square of the distance  $D$ . Therefore if neither the quantities of matter in  $A$  and  $B$ , nor the distance  $D$  be the same, the weight of  $A$  towards  $B$  will be as the quantity of matter in  $A$  directly, and as the quantity of matter in  $B$  directly, and as the square of the distance  $D$  inversely. Thus if  $A$  and  $B$  be numbers representing the quantities of matter in the bodies  $A$  and  $B$  respectively, the weight of  $A$  towards  $B$  at the distance  $D$  will be as  $\frac{AB}{D^2}$ , that is, the weight of  $A$  towards  $B$  at the distance  $D$  will be to the weight of  $a$  towards  $b$  at the distance  $d$  as the fraction  $\frac{AB}{D^2}$  is to the fraction  $\frac{ab}{d^2}$ .

Hence the weight of  $A$  towards  $B$  will be equal to the weight of  $B$  towards  $A$ , since both will be represented by the same quantity  $\frac{AB}{D^2}$ .

*Another Way of treating the Examples in the two last Articles.*

305. *If there be ever so many quantities, and these all heterogeneous to one another, we are at liberty to represent them by what number we please, or even all by unity itself, provided we take care to represent all other quantities of like kinds by proportionable numbers.* Thus I am at liberty to call any quantity of time I please 1, or any degree of velocity 1, or any quantity of space 1; but then I must take care to call a double time, or a double velocity, or a double space, by the number 2, and so on. This consideration suggests to us another way of treating the examples in the two last articles, somewhat different from the former; which, as it may be explained by a bare instance or two, I shall give the learner as follows:

In the first example we are taught that the space described by a body moving uniformly for any time, and with any velocity, is in numbers as the time and velocity multiplied together; which may also be demonstrated thus: suppose that a body moving uniformly in some known time called 1, and with some velocity called 1, shall describe a space which we will also call 1; then if in the time 1, and with the velocity 1, there be described the space 1, it is plain that in the time  $T$ , and with the velocity 1, there will be described the space  $T$ ; but if in the time  $T$ , and with the velocity 1, there be described the space  $T$ , then in the time  $T$ , and with the velocity  $V$ , there will be described the space  $VT$ , and that, let the quantities  $V$  and  $T$  be what they will; and therefore, in all cases, the space will be as  $T \times V$ .

Again,

Again, in the sixth example it was shewn that if any moving force as  $M$  be directly applied to any body whose quantity of matter is  $\mathcal{Q}$ , the velocity thereby produced will be as  $\frac{M}{\mathcal{Q}}$ : for a future demonstration whereof, let us suppose that some known force called 1, when applied to some quantity of matter called 1, will produce the velocity 1; then will the force 2 applied to the same quantity of matter 1 produce the velocity 2; but if the force 2 when applied to the quantity of matter 1 produce the velocity 2, then the same force 2 applied to a quantity of matter as 3 will produce a velocity equal to a third part of the former, to wit,  $\frac{2}{3}$ ; and for the same reason the force  $M$  applied to a quantity of matter as  $\mathcal{Q}$  will produce the velocity  $\frac{M}{\mathcal{Q}}$ ; and therefore this velocity will always be as  $\frac{M}{\mathcal{Q}}$ .

It is not impossible but that some of my less judicious readers may be inclined to think I have spun out this subject to too great a length: but I easily persuade myself that there are none who have thoroughly considered the very great usefulness and importance of this doctrine, especially in Mechanical and Natural Philosophy, but will readily acquit me of this charge; and the more so, because none that I know of have digested these matters into a system, or have written so distinctly upon them as the importance of the subject requires.

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T H E

E L E M E N T S   O F   A L G E B R A .

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B O O K   V I I I .   P A R T   I I .

*Of Prisms, Cylinders, Pyramids, Cones, and Spheres.*

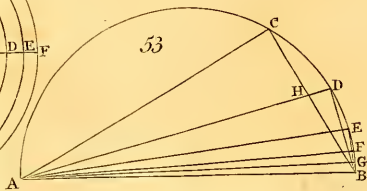
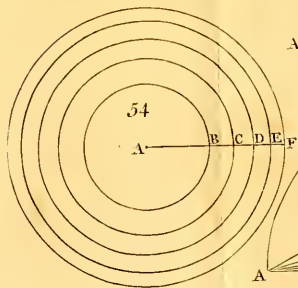
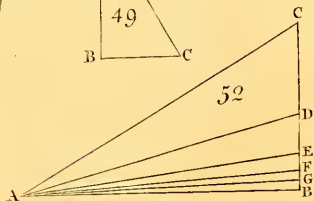
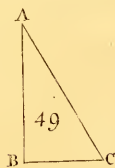
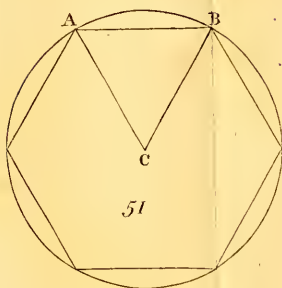
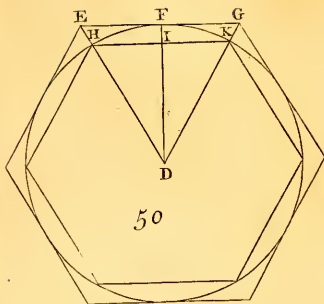
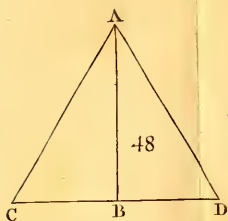
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**M**ANY of the following articles concerning the circle, sphere, and cylinder, are taken out of *Archimedes*, but demonstrated another way: and though they have no immediate relation to Algebra, yet as there are not many of them, and as they are a sort of supplement to *Euclid's* Geometry, I have been prevailed upon to insert them here, for the sake of those who cannot read *Archimedes*, and for the ease of those who can. Moreover, as *Euclid's* doctrine of solids is somewhat hard of digestion as it is delivered in the Elements, I have not scrupled to transfer some of the chief properties of cones and pyramids into this book, and to demonstrate them after a more easy and simple manner. And lastly, as the mensuration of the circle is absolutely necessary to the mensuration of the cylinder, cone, and sphere, I shall, before I enter upon the rest, explain what *Archimedes* has delivered upon that head.

A L E M M A .

to face pa. 356.

Pl. I.





A L E M M A.

340. *If in a right-angled triangle one of the acute angles be thirty degrees, or a third part of a right one, the opposite side will be equal to half the hypotenuse.* (Fig. 48.)

Let  $ABC$  be a right-angled triangle, right-angled at  $B$ , and let the angle  $BAC$  be 30 degrees; I say then that the opposite side  $BC$  will be half the hypotenuse  $AC$ .

For producing  $CB$  beyond  $B$  to  $D$ , so that  $BD$  may be equal to  $BC$ , and drawing  $AD$ , the two triangles  $ABC$  and  $ABD$  will be similar and equal; therefore the angle  $CAD$  will be 60 degrees, and the lines  $AC$  and  $AD$  will be equal; therefore the other two angles at  $C$  and  $D$  will be 60 degrees each, and the triangle  $ACD$  will be equilateral; therefore the line  $BC$ , which is the half of  $CD$ , will also be the half of  $AC$ . Q. E. D.

A L E M M A (Fig. 49, 50).

341. *Let  $ABC$  be a right-angled triangle, right-angled at  $B$ ; and supposing two similar and equilateral polygons, one to be circumscribed about a circle, and the other to be inscribed in it, let the angle  $BAC$  be equal to half the angle at the center subtended by a side of either polygon: I say then that  $AB$  will be to  $BC$  as the diameter of the circle to the side of the circumscribed polygon; and that  $AC$  will be to  $BC$  as the diameter of the circle is to the side of the inscribed polygon.*

Let  $D$  be the center of the circle, let  $EFG$  be a side of the circumscribed polygon, touching the circle in the point  $F$ , and let  $HIK$  be the side of a like polygon inscribed, and let  $HK$  and  $EG$  be supposed parallel, so as to subtend the same angle at the center. Draw the lines  $DHE$ ,  $DIF$ ,  $DKG$ ; then will the three triangles  $ABC$ ,  $DEF$ , and  $DHI$ , be similar, having the angles at  $B$ ,  $F$ , and  $I$ , right, and the angle  $BAC$  being equal to the angle  $EDF$  by the supposition; therefore

$AB$  will be to  $BC$  as  $DF$  to  $EF$ , or as  $2DF$  to  $EG$ , that is, as the diameter of the circle is to the side of the circumscribed polygon; and  $AC$  will be to  $BC$  as  $DH$  to  $HI$ , or as  $2DH$  to  $HK$ , that is, as the diameter of the circle is to the side of the inscribed polygon. *Q. E. D.*

If the angle  $BAC$  be a 48th part of a right one,  $AB$  will be to  $BC$  as the diameter of any circle is to the side of a regular polygon of 96 sides circumscribed about it, and  $AC$  will be to  $BC$  as the diameter is to the side of a like polygon inscribed. For if the line  $HIK$  be the side of an inscribed regular polygon of 96 sides, the arc  $HFK$  will be a 96th part of the whole circumference, or a 24th part of a quadrant, and the arc  $HF$  a 48th part of a quadrant; whence the angle  $EDF$  or  $HDI$  will be a 48th part of a right angle.

### A T H E O R E M.

342. *The circumference of every circle is somewhat more than three diameters. (Fig. 51.)*

Let  $AB$  be the side of a regular hexagon inscribed in a circle whose center is  $C$ , and draw  $AC$  and  $BC$ : then will the angle at  $C$  in the triangle  $ABC$  be 60 degrees, as containing a sixth part of the whole circumference; therefore since  $AC$  and  $BC$  are equal, the other two angles at  $A$  and  $B$  will be 60 degrees each; therefore the triangle  $ABC$  will be equiangular, and consequently equilateral; therefore  $AB$  will be equal to  $AC$ , and  $6AB$  to  $6AC$ ; but  $6AB$  is equal to the perimeter of the inscribed hexagon, and  $6AC$  is equal to three diameters; therefore the perimeter of a regular hexagon inscribed in a circle is equal to three times the diameter of that circle: whence it follows that the circumference of the circle itself will be somewhat more than three diameters. *Q. E. D.*

## A T H E O R E M.

343. *If the diameter of a circle be called 1, the circumference will be somewhat less than  $3\frac{10}{70}$  and somewhat greater than  $3\frac{10}{71}$ .*

*The Demonstration of the first Part. (Fig. 52.)*

Let  $ABC$  be a right angle, in which inscribe the lines  $AC, AD, AE, AF, AG$  in the manner following: make the angle  $BAC$  a third part of a right one,  $BAD$  a 6th part,  $BAE$  a 12th part,  $BAF$  a 24th part, and  $BAG$  a 48th part: then will  $AC$  be double of  $BC$  by the 340th article, and  $AB$  will be to  $BG$  as the diameter of any circle is to the side of a regular polygon of 96 sides circumscribed about it by the 341st article. Moreover, as the line  $AD$  bisects the angle  $BAC$ , we shall have as  $AB$  to  $AC$  so  $BD$  to  $DC$  by the third of the sixth book of the Elements; and by art. 330\*,  $AB+AC$  is to  $AB$  as  $BC$  is to  $BD$ ; and by permutation,  $AB+AC$  is to  $BC$  as  $AB$  is to  $BD$ : therefore if  $BC$  be divided into any number of equal parts, how many soever of these parts are contained in the sum of the lines  $AB$  and  $AC$ , the same number of like parts of  $BD$  will be contained in the line  $AB$  alone; as if  $BC$  be divided into 10000 equal parts, and the sum  $AB+AC$  contains 37320 of those parts, then if the line  $BD$  be divided into 10000 equal parts, the line  $AB$  alone will contain 37320 of them. After the same manner it may be demonstrated, that whatever parts of  $BD$  are contained in the sum of the lines  $AB, AD$ , the same number of like parts of  $BE$  will be contained in  $AB$  alone, and so on: whence we have the following process.

1st. Let  $BC$  be divided into 10000 equal parts, or (which is the same thing) let  $BC$  be called 10000;

\* See the Quarto Edition, p. 539.

then will  $AC$  be 20000, and consequently  $AB$  will be greater than 17320, and  $AB + AC$  will be greater than 37320.

2dly, Therefore if  $BD = 10000$ ,  $AB$  will be greater than 37320,  $AD$  greater than 38636, and  $AB + AD$  greater than 75956.

3dly, Therefore if  $BE = 10000$ ,  $AB$  will be greater than 75956,  $AE$  greater than 76611, and  $AB + AE$  greater than 152567.

4thly, Therefore if  $BF = 10000$ ,  $AB$  will be greater than 152567,  $AF$  greater than 152894, and  $AB + AF$  greater than 305461.

5thly, Therefore if  $BG = 10000$ ,  $AB$  will be greater than 305461: therefore *e converso*, if  $AB$  be supposed equal to 305461,  $BG$  will be less than 10000: but it was shewn before that  $AB$  is to  $BG$  as the diameter of any circle is to the side of a regular polygon of 96 sides circumscribed about that circle; therefore if the diameter of any circle be called 305461, the side of such a polygon will be less than 10000, and the whole perimeter less than 960000; therefore the perimeter of such a polygon will be less than the product

of the diameter multiplied into  $3 \frac{10}{70}$  or  $\frac{22}{7}$ : for

$305461 \times \frac{22}{7} = 960020 \frac{2}{7}$ : therefore if the diameter

of any circle be called 1, the perimeter of a regular polygon of 96 sides circumscribed about it will be

less than  $3 \frac{10}{70}$ ; but the circumference of every circle

is less than the perimeter of any polygon circumscribed about it; therefore the circumference of the circle will

still be less than  $3 \frac{10}{70}$ . Q. E. D.

*The Demonstration of the second Part. (Fig. 53.)*

Let  $ACDEFG$  be a semicircle whose diameter is  $AB$ , and in this semicircle let the lines  $AC$ ,  $AD$ ,  $AE$ ,  $AF$ ,  $AG$ , be inscribed in the manner following: make the angle  $BAC$  a third part of a right one,  $BAD$  a sixth part,  $BAE$  a 12th part,  $BAF$  a 24th part, and  $BAG$  a 48th part, and join  $BC$ ,  $BD$ ,  $BE$ ,  $BF$ ,  $BG$ ; then will  $AB$  be double of  $BC$ , and  $AB$  will be to  $BG$  as the diameter of any circle is to the side of a regular polygon of 96 sides inscribed. Let  $AD$  cut  $BC$  in  $H$ ; and by the demonstration of the first part of this theorem,  $AC \perp AB$  will be to  $CB$  as  $AC$  to  $CH$ , since by the construction the line  $AH$  bisects the angle  $BAC$ : but the triangles  $ACH$  and  $ADB$  are similar, having the angles at  $C$  and  $D$  right, as being in a semicircle, and the angle  $CAH$  being equal to the angle  $DAB$ ; therefore  $AC$  will be to  $HC$  as  $AD$  to  $BD$ : but it was before demonstrated, that as  $AC$  is to  $HC$  so is  $AB \perp AC$  to  $BC$ ; therefore as  $AB \perp AC$  is to  $BC$  so is  $AD$  to  $BD$ ; and whatever parts of  $BC$  are contained in the sum of the lines  $AB$ ,  $AC$ , the same number of like parts of  $BD$  will be contained in the line  $AD$  alone: whence the following process.

1st, Let  $BC=10000$ ; then will  $AB=20000$ ,  $AC$  will be less than 17321, and  $AB \perp AC$  will be less than 37321.

2dly, Therefore if  $BD=10000$ ,  $AD$  will be less than 37321,  $AB$  will be less than 38638, and  $AB \perp AD$  will be less than 75959.

3dly, Therefore if  $BE=10000$ ,  $AE$  will be less than 75959,  $AB$  will be less than 76615, and  $AB \perp AE$  will be less than 152574.

4thly, Therefore if  $BF=10000$ ,  $AF$  will be less than 152574,  $AB$  will be less than 152902, and  $AB \perp AF$  will be less than 305476.

5thly, Therefore if  $BG=10000$ ,  $AG$  will be less than 305476, and  $AB$  will be less than 305640; therefore

therefore *e converso*, if  $AB$  be equal to 305640,  $BG$  will be greater than 10000: but  $AB$  is to  $BG$  as the diameter of any circle is to the side of a regular polygon of 96 sides inscribed in it; therefore if the diameter of any circle be 305640, the side of such an inscribed polygon will be greater than 10000, and its perimeter greater than 960000; therefore the perimeter of such a polygon will be greater than the product of the diameter multiplied into  $3 \frac{10}{71}$  or  $\frac{223}{71}$ : for  $305640 \times \frac{223}{71} = 959968 - \frac{8}{71}$ : therefore if the diameter of a circle be called 1, the perimeter of a regular polygon of 96 sides inscribed in it will be greater than  $3 \frac{10}{71}$ : but the circumference of every circle is greater than the perimeter of any inscribed polygon; therefore the circumference of this circle will be greater still than  $3 \frac{10}{71}$ . *Q. E. D.*

Thus then if the diameter of a circle be called 1, the circumference must lie between these two very narrow limits, to wit,  $3 \frac{10}{70}$  and  $3 \frac{10}{71}$ : the whole difference of these limits is but  $\frac{1}{497}$ , and therefore, by this method, the circumference of a circle is determined to a 497th part of the diameter.

*The most compendious Way of obtaining the Numbers in the last Article.*

344. If any one has a mind to examine the foregoing calculations, it may not be amiss to let him know, that the hypotenuses of the triangles  $ABD$ ,  $ABE$ ,  $ABF$ , and  $ABG$  (Fig. 52, 53), may be computed without either squaring the greater leg, or extracting

tracting the more considerable part of the square root. As if  $AD$  (*Fig. 52*), the hypotenuse of the triangle  $ABD$ , in the first part be required, having given  $AB$  37320 and  $BD$  10000, the method I use is as follows :

1st, Whatever the hypotenuse  $AD$  may be, this is certain, that the greater leg  $AB$  will be equal to a considerable part of it ; and therefore if  $AD$  be to be found by a series, as is usual in extracting the square root, it will be proper to make  $AB$  the first term ; and hence it is that I call  $37320 = AB$  my first root. Again, as the square of  $AD$  is to exceed the square of  $AB$  by the square of  $BD$ , that is, by 100000000 ; this number I call my first resolvend, and then doubling my first root, the product 74640 I call my first divisor, and so am prepared for the following operation.

2dly, Thus prepared, I divide my first resolvend by my first divisor, and the first figure of the quotient (for I am concerned for no more at present) I find to be 1, which, as it comes out of the place of thousands, signifies 1000 ; this number therefore 1000 I add to my first root, and so have 38320 for a more correct or second root. The same number 1000 I add also to my first divisor, and then multiplying the sum 75640 by 1000, the number that was added, I subtract the product 75640000 from my first resolvend, and there remains 24360000 ; this I call my second resolvend, and the double of my second root, to wit, 76640, I call my second divisor, and so proceed to the next operation.

3dly, Now I divide my second resolvend by my second divisor, and the first figure of the quotient is 3, which, as it comes out of the place of hundreds, signifies 300 ; therefore I add 300 to my second root, and so have 38620 for my third root : the same number 300 I also add to my second divisor, and the sum 76940 I multiply by 300, and the product is 23082000, which, being subtracted from my second resolvend, leaves me 1278000 for a third resolvend,  
and

and the double of my third root; to wit, 77240, I have for my third divisor.

4thly, I divide my third resolvend by my third divisor, and the first figure of the quotient is 1, which signifies 10; therefore I add 10 to my third root, and so have a fourth root 38630: moreover adding 10 to my third divisor, the sum is 77250, which being multiplied by 10, and the product 772500 being subtracted from the third resolvend, leaves 505500 for the fourth resolvend, and the double of my fourth root, to wit, 77260, makes a fourth divisor.

5thly and lastly, I divide my fourth resolvend by my fourth divisor, and the nearest quotient too little is 6; therefore I add 6 to my fourth root, and so have a fifth root, to wit, 38636, which is the nearest root less than the true that can be expressed in whole numbers: therefore the hypotenuse *AD* is greater than 38636.

The reason of these operations will not be difficult to any one who thoroughly understands the foundation of the common method of extracting the square root.

*Van Ceulen's Numbers expressing the Circumference of a Circle whose Diameter is 1.*

345. This method of *Archimedes* is capable of being pursued to any degree of exactness required: nay *Ludolf Van Ceulen* has computed the circumference of a circle to no fewer than 36 places, upon a supposition that the diameter is unity. His numbers expressing this circumference are,

3.1415 9265 3589 7932 3846 2643 3832 7950 288 +.  
But since the invention of fluxions by its great author *Sir Isaac Newton*, he (*Sir Isaac*) has from this method drawn serieses almost infinitely more expeditious than the bisections of *Archimedes* or *Van Ceulen*, whereby the circumference of a circle may be computed to 12 or 13 places in little more than half an hour's time,

Art. 345. 346. *Why the Circle cannot be squared, &c.* 365

as Doctor *Halley*\* from his own experience assures us.

*Note*, that *Metius's* proportion of the diameter of a circle to the circumference is as 113 to 355, the most accurate of any in such small numbers. See Schol. 1. in art. 179 †.)

*Why the Circle cannot be squared geometrically.*

346. If, having given the diameter or semidiameter of any circle, a right line could be found exactly equal to the circumference, whether such a line could be expressed by numbers or not, the circle might be squared as well as any right-lined figure whatever, that is, a square might be constructed whose area would be equal to that of the circle, which I thus demonstrate.

Let  $2r$  represent the diameter of any circle, and  $2c$  the circumference; then will  $rc$ , the product of the *radius* into the semicircumference, be its area, by cor. 4 in art. 311 ‡. Let now  $x$  be the side of a square whose area is equal to that of the circle, and we shall have  $xx = rc$ ; whence  $x$  will be a mean proportional between  $r$  and  $c$ , and may be found by the 13th of the sixth book of the *Elements*. But it is impossible upon *Euclid's postulata*, having given the diameter or semidiameter of any circle, to find a right line exactly equal to the circumference; and therefore the circle cannot be squared upon the same foundation on which we are taught to square all right-lined figures; and hence it is that we say, the circle cannot be squared geometrically. But as the three *postulata* of *Euclid*, how simple soever they may appear in theory, are never a one of them capable of being perfectly executed, but that errors must necessarily arise in drawing and producing lines, in taking the distances of points, &c.; and as from these errors others must necessarily arise in subsequent

\* Dr. Halley's series is transformed into two others, more convenient for numerical calculation, in my *Mathematical Essays*, p. 120. J. H. † See the Quarto Edition, p. 282. ‡ Ib. p. 504. operations;

operations ; and lastly, as the circumference of a circle may be had from the diameter in numbers, to any assignable degree of exactness, it follows that in practice, a circle is as capable of being squared as any other figure whatever that is not a square.

*Corollaries drawn from Art. 343.*

347. From the 343d article may be deduced several corollaries, some of the most useful whereof are inserted here as follows :

1st, *The diameter of every circle is to the circumference as 7 to 22 nearly : for 1 is to 3  $\frac{10}{70}$  or  $\frac{22}{7}$  as 7 to 22.*

2d, *If  $d$  be the diameter of any circle, its area will be  $\frac{11dd}{14}$  : for as 7 is to 22, so is  $d$  the diameter to  $\frac{22d}{7}$  the circumference ; and if  $\frac{d}{2}$  the radius be multiplied into  $\frac{11d}{7}$  the semicircumference, the pro-*

*duct  $\frac{11dd}{14}$  will be the area, by corollary 4 in art. 311\*.*

3d, *Hence we have a ready way, having the diameter of any circle given, to find its area, and vice versa, without the mediation of the circumference, by saying, as 14 is to 11, so is the square of the given diameter to the area sought. But if the area be given, in order to find the diameter, the proportion must be reversed, by saying as 11 is to 14, so is the given area to a fourth, which fourth number will be the square of the diameter, and its square root the diameter itself.*

4th, *Arguing as in the two last corollaries, If the diameter of a circle be to the circumference as  $a$  to  $b$ , then the square of the diameter of any circle will be to its area as  $4a$  to  $b$  ; and vice versa, the area will be to the square of the diameter as  $b$  to  $4a$ .*

\* See the Quarto Edition, p. 504.

5th, *The circumferences of all circles are as their diameters or semidiameters, and their areas as the squares of the diameters or semidiameters.* For if  $d$  and  $e$  be the diameters of two circles, their circumferences will be  $\frac{22d}{7}$  and  $\frac{22e}{7}$ ; and  $\frac{22d}{7}$  is to  $\frac{22e}{7}$  (dropping the common denominator 7, and the common factor 22) as  $d$  to  $e$ . Again, the area of the circle whose diameter is  $d$  is  $\frac{11dd}{14}$  as in the second corollary; and for the same reason, the area of the other circle whose diameter is  $e$  is  $\frac{11ee}{14}$ ; and  $\frac{11dd}{14}$  is to  $\frac{11ee}{14}$  as  $dd$  to  $ee$ ; therefore the circumferences of all circles are as their diameters, and their areas as the squares of their diameters. And since the halves of all quantities are as the wholes, and the squares of the halves as the squares of the wholes, it follows also that the circumferences of circles are as their semidiameters, and their areas as the squares of the semidiameters.

6th, *If there be three circles such, that the sum of the squares of the semidiameters of two of them is equal to the square of the semidiameter of the third; I say then that the areas of the two first circles put together will be equal to the area of the third.* For let  $a, b, c$ , represent the semidiameters of the three circles, and let  $a^2 + b^2 = c^2$ : since then the semidiameter of the first circle is  $a$ , the diameter will be  $2a$ , and the square of the diameter  $4aa$ : but as 14 is to 11 so is  $4aa$  to  $\frac{44a^2}{14}$  or  $\frac{22a^2}{7}$ ; therefore the area of the first circle will be  $\frac{22a^2}{7}$  by the third corollary; and for the same reason, the areas of the other two circles will

will be  $\frac{22b^2}{7}$  and  $\frac{22c^2}{7}$ : but  $a^2 + b^2 = c^2$  *ex hypothesi*:

therefore  $\frac{22a^2}{7} + \frac{22b^2}{7} = \frac{22c^2}{7}$ .

*N. B.* This last corollary is not demonstrated in the 31<sup>st</sup> of the sixth book of the Elements, as some may imagine, that demonstration not reaching farther than right-lined figures.

The following easy problems may serve to shew some uses of the foregoing corollaries.

### P R O B L E M I.

348. *To find the proportion between the diameter of any circle and the side of an equal square.*

Call this diameter 1, and by the second corollary in the foregoing article, the area of this circle will be

$\frac{11}{14}$  nearly; and the side of a square whose area is

$\frac{11}{14}$  will be  $\sqrt{\frac{11}{14}}$ : therefore the diameter of any

circle is to the side of an equal square as 1 to

$\sqrt{\frac{11}{14}}$ . But because this fraction  $\frac{11}{14}$ , though it

serves well enough for common purposes, is not accurately true, and because its square root cannot be accurately expressed in numbers neither, to reduce the error (for there must be an error) to a more simple point, let  $c$  be the circumference of a circle whose diameter is 1; and the area of such a circle, that is, the product of the *radius* into the semicir-

cumference, will be  $\frac{1}{2} \times \frac{c}{2} = \frac{c}{4}$ ; and the side of

an equal square will be  $\sqrt{\frac{c}{4}}$ : but, according to *Van*

*Ceulen*,  $c = 3.1415926536$ , and  $\frac{c}{4} = .7853981634$ ,

and

and  $\sqrt{\frac{c}{4}} = .88623$ ; therefore the diameter of a circle is to the side of an equal square as 1 to .88623. or as 100000 is to 88623: suppose the proportion to be that of 100000 to 88625, which makes but an error of 2 in the fifth place, and then multiplying both terms by 8, the proportion will be that of 800000 to 709000, or of 800 to 709; therefore *As 800 is to 709, so is the diameter of any circle to the side of an equal square, nearly true to five places.*

N. B. If the diameter of a circle be 9 yards, the side of an equal square found as above will not err an hundredth part of an inch.

#### PROBLEM 2.

349. *To find the semidiameter of a circle that will comprehend within its circumference the quantity of an acre of land.*

An acre of land contains 4840 square yards, and therefore this must be the area of our circle. Say then, as 11 to 14, so 4840 to 6160; and this last number will be the square of the diameter, by the third corollary in art. 347; whence the diameter itself will be 78.486 yards, and the semidiameter 39.243 yards, that is 39 yards  $8\frac{3}{4}$  inches nearly.

#### PROBLEM 3.

350. *Let a string of a given length be disposed into the form of a circle: It is required to find the area of this circle.*

Let  $c$  be the length of the string, and consequently the circumference of the circle made by it, and the diameter of the circle will be  $\frac{7c}{22}$ , the semidiameter

$\frac{7c}{44}$ , and the area  $\frac{7cc}{88}$ . Suppose now this string to be disposed into the form of a square, and the side

A a of

of this square would be  $\frac{c}{4}$ , and its area  $\frac{cc}{16}$ , and the area of the square would be to the area of the circle as  $\frac{cc}{16}$  is to  $\frac{7cc}{88}$ , that is, as  $\frac{1}{16}$  is to  $\frac{7}{88}$ , or as 11 to 14: therefore, *As 11 is to 14, so is the area comprehended by the string when in form of a square, to the area comprehended by the same string when in form of a circle.*

Q. E. I.

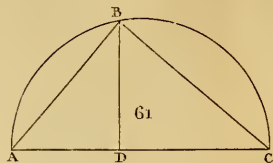
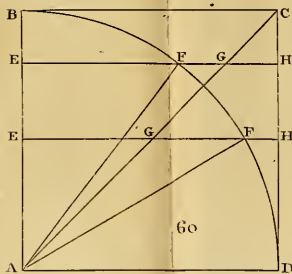
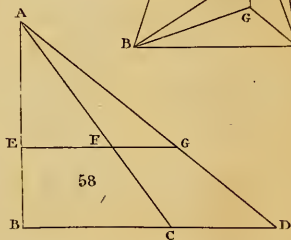
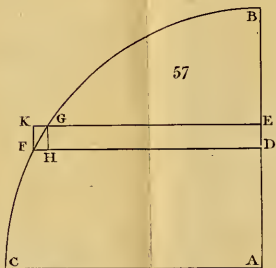
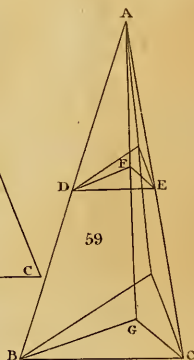
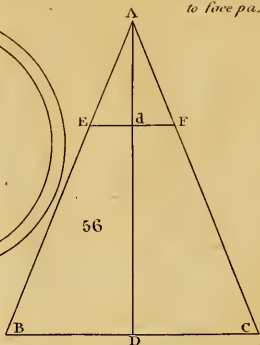
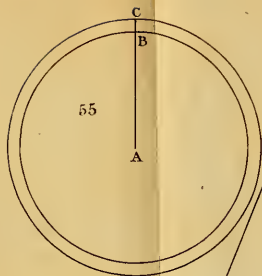
*N. B.* By the answer here given it appears, that if  $c$  be the circumference of any circle, its area will be  $\frac{7cc}{88}$ ; and consequently that *As 88 is to 7, so is the square of the circumference of any circle to its area nearly.*

#### P R O B L E M 4.

351. *It is required to divide a given circle into any number of equal parts by means of concentric circles drawn within it. (Fig. 54.)*

Let  $A$  be the center, and  $AF$  be the semidiameter of the circle given, and let it be required to divide the area of this circle into five equal parts by means of four concentric circles described within the former, and cutting the line  $AF$  in the points,  $B, C, D, E$ , that is, let the area of the circle  $AB$ , and the areas of the annuli  $BC, CD, DE$ , and  $EF$ , be supposed all equal; then the circle  $AB$  will be  $\frac{1}{5}$  the whole, the circle  $AC$   $\frac{2}{5}$ , the circle  $AD$   $\frac{3}{5}$ , &c.; and the area of the circle  $AF$  will be to the area of the circle  $AB$  as 5 to 1: but the area of the circle  $AF$  is to the area of the circle  $AB$  as the square of the semidiameter  $AF$  is to the square of the semidiameter  $AB$ , by cor. 5. in art. 347; therefore  $AF^2$  is to  $AB^2$  as 5 to 1: but  $AF^2$  is given by the supposition; therefore  $AB^2$ , and consequently  $AB$  the semidiameter of the inmost circle is given. In like manner  $AF^2$  is to  $AC^2$  as 5

to





to 2; whence  $AC$  the semidiameter of the next concentric circle is given; and so of the rest. *Q. E. I.*

# PROBLEM 5.

352. *Whoever makes a tour round the earth, must necessarily take a larger compass with his head than with his feet: The question is, how much larger?*

Let  $A$  (Fig 55.) represent the center of the earth,  $AB$  its semidiameter,  $BC$  the traveller's height,  $AC$  the semidiameter of the circle described by his head: let also  $b$  represent the circumference of the circle whose semidiameter is  $AB$ , and  $c$  the circumference of the circle whose semidiameter is  $AC$ , and  $c - b$  will be the difference we are now enquiring into, which may be thus determined.

By the fifth corollary in art. 347,  $AC$  is to  $AB$  as  $c$  is to  $b$ ; and by division of proportion,  $BC$  is to  $AB$  as  $c - b$  is to  $b$ ; and alternately,  $BC$  is to  $c - b$  as  $AB$  is to  $b$ . Let  $d$  be the circumference of a circle whose semidiameter is  $BC$ , and  $BC$  will be to  $d$  also as  $AB$  to  $b$ ; therefore  $BC$  is to  $d$  as  $BC$  is to  $c - b$ ; therefore,  $c - b = d$ ; that is, *The traveller's head will pass through more space than his feet by the circumference of a circle whose semidiameter is his own length*: as if the man be 6 feet high, his head will travel farther than his heels by 37 feet  $8\frac{4}{7}$  inches nearly, and that whether the semidiameter  $AB$  be greater or less, or nothing at all.

# PROBLEM 6.

353. *It is required, having given the depth and the diameter of the base of any cylindrical vessel, to find its content in ale gallons.*

Here it must be observed, that in the mensuration of a solid, all its dimensions must be taken in the same kind of measure: as, if any one dimension be taken in inches, the rest must be taken so too, and then the number representing the content of any

solid will be the number of cubic inches to which that solid is equivalent.

Let then  $a$  be the given altitude of the cylindrical vessel to be measured,  $d$  the diameter of its base, and by the second corollary in art. 347,  $\frac{11dd}{14}$  will give a number of square inches equal to the base, and this area multiplied into the altitude  $a$ , will give  $\frac{11add}{14}$ , a number of cubic inches equal to the solid content of the vessel: but 282 cubic inches constitute an ale gallon; and therefore if  $\frac{11add}{14}$ , the solid content of the vessel, be divided by 282, the quotient to wit,  $\frac{11add}{3948}$ , will be the number of gallons therein contained. Instead of 3948 put 3949, which will make no considerable difference in so great a denominator, and the fraction  $\frac{11add}{3949}$  (dividing both the numerator and denominator by 11) will be reduced to  $\frac{add}{359}$ : whence the following canon:

*Having taken both the depth, and the diameter of the base in inches, multiply the square of the diameter into the depth of the vessel, and the product divided by 359 will give the number of gallons required.*

*N. B.* This substitution of 3949 instead of 3948 corrects about  $\frac{7}{10}$  of the other that would otherwise have been committed in supposing the square of the diameter of the base to be its area as 14 to 11.

# PROBLEM 7.

354. *To measure a frustrum of a cone, whose perpendicular altitude and the diameters of the two bases are given.*

*N. B.*

N. B. By a frustum of a cone I mean a cone having its top cut off by a plane parallel to the base.

Let the isosceles triangle  $ABC$  (Fig 56.) represent the section of a cone made through its axis  $AD$ , and let  $EF$  be any line parallel to the base  $BC$ , cutting  $AB$  in  $E$ ,  $AC$  in  $F$ , and the axis  $AD$  in  $d$ ; then will the trapezium  $BEFC$  be the section of a frustum of this cone whose perpendicular altitude is  $Dd$ . Call  $BC$ , the diameter of the greater base,  $g$ ;  $EF$ , the diameter of the lesser base,  $l$ ; and  $Dd$ , the height of the frustum,  $b$ : call also  $AD$ , the unknown altitude of the whole cone,  $x$ ; and consequently  $Ad$ , the altitude of the cone to be cut off,  $x - b$ ; and from the similar triangles  $ABC$ ,  $AEF$ , we have this proportion;  $AD$  is to  $Ad$  as  $BC$  is to  $EF$ , that is, according to our notation,  $x$  is to  $x - b$  as  $g$  to  $l$ ; whence, by multiplying extremes and means, we have  $gx - gb = lx$ , and  $x$ , or the altitude of the cone, equal to  $\frac{gb}{g-l}$ . In like manner if from the value of  $x$  we subtract  $b$ , the altitude of the frustum, we shall find  $Ad$ , or the height of the cone to be cut off, equal to  $\frac{lb}{g-l}$ . Now the solid content of every cone is found by multiplying the base into a third part of its altitude; therefore since the base of the cone  $ABC$  is  $\frac{11gg}{14}$ , and its altitude  $\frac{gb}{g-l}$ , its solid content will be  $\frac{g^3}{g-l} \times \frac{b}{3} \times \frac{11}{14}$ : in like manner the solid content of the cone  $AEF$  will be  $\frac{l^3}{g-l} \times \frac{b}{3} \times \frac{11}{14}$ : subtract the latter cone from the former, and there will remain the solid content of the frustum  $BEFC$

*BEFC* equal to  $\frac{g^3 - l^3}{g - l} \times \frac{b}{3} \times \frac{11}{14}$  : but the fraction  $\frac{g^3 - l^3}{g - l}$  may be reduced to an integer by dividing the numerator by the denominator, and the quotient will be  $gg + gl + ll$  ; therefore the solid content of the frustum *BEFC* will now be expressed thus,  $\overline{gg + gl + ll} \times \frac{b}{3} \times \frac{11}{14}$ . Whence we have the following canon :

*Add the squares and the rectangle of the two given diameters together ; multiply the sum into a third part of the given altitude, and the product will be the frustum of a pyramid of the same height having square bases whose sides are equal to the two diameters given ; and as 14 is to 11, so will this frustum be to the frustum sought.* Q. E. I.

*N. B.* 1<sup>st</sup>, Since a cone differs nothing from a frustum of a cone whose lesser base is equal to nothing, if *l* be made equal to nothing in the foregoing canon, it ought to give the solid content of a cone whose height is *b*, and the diameter of whose base is *g* : and so it will ; for then  $\overline{gg + gl + ll} \times \frac{b}{3} \times \frac{11}{14}$

Becomes  $\frac{11gg}{14} \times \frac{b}{3}$ .

2<sup>dly</sup>, Since a cylinder may be considered as a frustum of a cone whose bases are equal, if *l* be made equal to *g* in the foregoing canon, it ought to give the solid content of a cylinder whose height is *b*, and the diameter of whose base is *g* : and so we find it will ; for  $\overline{gg + gl + ll} \times \frac{b}{3} \times \frac{11}{14}$  in this case becomes

$3gg \times \frac{b}{3} \times \frac{11}{14} = \frac{11}{14} ggb$ .

3<sup>dly</sup>,

3dly, If the lesser base of the frustum be supposed to be increased till it becomes equal to the greater; and if, on the other hand, the greater base be supposed to be diminished till it becomes equal to that which was the lesser base before, the solid content of the frustum will be the same as at the first; and therefore, if the foregoing canon be just, it ought not to be altered by changing the quantities  $g$  and  $l$  one for the other: which is true; for  $gg + gl + ll$  by this means only becomes  $ll + lg + gg$ , which is the same quantity.

In solving this last problem it is taken for granted that every cone is the third part of a cylinder having the same base and height; which may safely be done, since *Euclid* has demonstrated it in the 10th of the twelfth book of the *Elements*: but because *Euclid's* doctrine of solids is not so easy to the imaginations of young beginners, I shall (in another place) give another demonstration of this proposition, independently of any thing that has here been said.

LEMMA. (Fig. 57.)

355. Let  $ABC$  be any curvilinear space comprehended between two straight lines  $AB$  and  $AC$  at right angles to each other, and a curve as  $BC$  concave towards  $AB$ ; and from any two points  $D$  and  $E$  in the line  $AB$  let the lines  $DF$  and  $EG$  be drawn parallel to the base  $AC$ , and terminating in the curve at the points  $F$  and  $G$ , and compleat the parallelogram  $DEGH$ : then it is plain that the curvilinear space  $DEGF$  will be greater than the parallelogram  $DEGH$ . But what I here propose to demonstrate is, that if the line  $EG$  be supposed to move towards  $DF$  in a position always parallel to itself, and at last to coincide with  $DF$ , the nearer  $EG$  approaches to  $DF$ , the nearer will the ratio of the curvilinear space  $DEGF$  to the parallelogram  $DEGH$  approach towards a ratio of equality, and that it will at last terminate in a ratio of equality when  $EG$  coincides with  $DF$ .

For, completing the parallelogram  $GHEK$ , the parallelogram  $DEKF$  will be to the parallelogram  $DEGH$  upon the same base as  $DF$  is to  $EG$ ; therefore the curvilinear space  $DEGF$  will be to the parallelogram  $DEGH$  in a less ratio than that of  $DF$  to  $EG$ ; that is, though that space be greater than this parallelogram, yet the ratio of the former to the latter is a less ratio than that of  $DF$  to  $EG$ : but the nearer the line  $EG$  approaches towards  $DF$ , the nearer will the ratio of  $DF$  to  $EG$  approach towards a ratio of equality, and it will at last become a ratio of equality when  $EG$  coincides with  $DF$ ; therefore *a fortiori*, the ultimate ratio of the curvilinear space  $DEGF$  to the parallelogram  $DEGH$  will be a ratio of equality.

Hence it follows, that if we suppose the line  $AB$  to be divided into a certain number of parts, such as  $DE$ , and these parts to be made the bases of so many inscribed parallelograms, such as is the parallelogram  $DG$ , the more there are of these parallelograms, the nearer will the sum of all the curvilinear spaces  $DEGF$ , that is, the whole curvilinear space  $ABC$ , approach towards the sum of all the inscribed parallelograms; and if we suppose the bases of these parallelograms to be diminished, and so their number to be augmented ad infinitum, they will make up the whole curvilinear space  $ABC$ ; so that the space  $ABC$  will be to the sum of all the inscribed parallelograms ultimately in a ratio of equality. For, setting aside the parts infinitely near the point of intersection  $B$ , which will be of no consequence in the account, let the parallelogram  $DEGH$  be that which, of all the rest, differs most from its correspondent curvilinear space  $DEGF$ ; and the consequence will be that the curvilinear space  $ABC$  will be to the sum of all the inscribed parallelograms in a less ratio than that of the space  $DEGF$  to the space  $DEGH$ : but even this ratio becomes at last a ratio of equality, when  $DE$  is infinitely small, by this lemma: whence it follows *a fortiori*, that the ultimate ratio of the curvilinear space  $ABC$  to the sum of all the inscribed parallelograms will be a ratio of equality.

vilinear space  $ABC$  to the sum of all the inscribed parallelograms will be a ratio of equality.

I thought myself obliged to demonstrate this proposition, because I have known it objected, that though the difference between any particular parallelogram and its correspondent curvilinear space be allowed to be infinitely small when the common base is so; yet how do we know but that an infinite number of these differences may amount to a finite quantity? and if so, then the whole curvilinear space cannot be said to be to the sum of all the inscribed parallelograms in a ratio of equality. To this I answer, that it must be the business of Geometry to determine whether an infinite number of these infinitely small differences amount to a finite quantity or not; and by the demonstration here given it appears that they do not, but that the sum of all these differences actually diminishes as their number increases, and at last comes to nothing when their number is infinite.

A L E M M A. (Fig. 57.)

356. *Supposing the line  $AB$  still to keep its place, let us imagine the whole space  $ABC$  to turn round it, so as to describe or generate a solid whose axis is  $AB$ , and the semidiameter of whose base is  $AC$ , and the inscribed parallelograms will at the same time by their common motion describe so many thin cylindric laminæ, which, taken all together, will be less than the solid generated by the space  $ABC$ ; but, the more they are in number, the nearer they will approach to it, and their circular edges will at last terminate in the curve surface of the solid when their number is infinite.*

For, completing the parallelogram  $GHFK$  as before, the lamina generated by the parallelogram  $DK$  will be to the lamina generated by the parallelogram  $DG$  as the square of  $DF$  is to the square of  $EG$ , all circles being as the squares of their semidiameters; therefore the lamina generated by the curvilinear space  $DEGF$  will be to the lamina generated by the parallelogram

lelogram  $DG$  in a less ratio than that of  $DF^2$  to  $EG^2$  : but when  $D$  and  $E$  coincide,  $DF$  will be equal to  $EG$ , and the square of  $DF$  to the square of  $EG$  ; therefore, in this case, every particular cylindric *lamina* will be the same with a correspondent *lamina* of the solid ; and *componendo*, all the cylindric *laminæ* will constitute the solid itself. This may also be further evident by applying the demonstration in the last lemma to this case. Therefore we need not scruple to suppose round solids, generated after the same manner as this is, to be made up of an infinite number of infinitely thin cylindric *laminæ* : Nay, even the cone itself may be considered as being so constituted : for if we suppose  $BC$  to be a straight line instead of a curve, the reasoning of the last article and this will equally succeed ; in which case, the space  $ABC$  will be a triangle, and the figure generated a cone.

If a solid be made up of a finite number of prismatic or cylindric *laminæ*, decreasing in their diameters as they are farther and farther distant from the base, the surface of such a solid must necessarily ascend by steps : but the thinner these elementary *laminæ* are (supposing their thinness to be compensated by a greater number), the narrower and the shallower these steps will be, so as to terminate at last in a regular surface when their number is infinite.

### A T H E O R E M.

357. All isosceles cones of equal heights are as their bases ; that is, the solid content of any one isosceles cone is to the solid content of any other of an equal height, as the base of the former cone is to the base of the latter.

Note, that by an isosceles cone I mean a cone whose base is a circle, and whose vertex is every where equally distant from the circumference of the base ; and by an isosceles pyramid is meant a pyramid having a regular polygon for its base, and whose vertex is equally distant from all the angles of that polygon : lastly, by isosceles

*prisms and cylinders are meant such as have equal and regular polygons and circles for their bases, and are so constituted, that a right line joining the centers of their two bases is perpendicular to them.*

Let  $ABC$  (Fig. 58.) be a right-angled triangle at  $B$ , and producing the base  $BC$  out of  $D$ , join  $AD$ ; let also the line  $EFG$  be drawn any where within the triangle parallel to the base  $BCD$ , so as to cut  $AB$  in  $E$ ,  $AC$  in  $F$ , and  $AD$  in  $G$ : then will  $EF$  be to  $BC$  as  $EG$  is to  $BD$ , since both are as  $AE$  to  $AB$  by similar triangles: therefore, if they be taken alternately,  $EF$  will be to  $EG$  as  $BC$  to  $BD$ , and  $EF^2$  to  $EG^2$  as  $BC^2$  to  $BD^2$ . This being allowed, let the triangle  $ABD$  be supposed to turn round the fixed side  $AB$ , so as to generate a cone whose *axis* is  $AB$ ; then will the triangle  $ABC$  generate another cone having the same common altitude  $AB$ . Let both these cones be considered as constituted of an infinite number of infinitely thin cylindric *laminæ*, and let  $EF$  represent indifferently the semidiameter of any one of these *laminæ* belonging to the cone  $ABC$ ; then will  $EG$  be the semidiameter of a correspondent *lamina* belonging to the cone  $ABD$ ; and the *lamina* whose semidiameter is  $EF$  will be to the *lamina* whose semidiameter is  $EG$ , having the same thickness, as  $EF^2$  is to  $EG^2$ , or (according to what is already demonstrated) as  $BC^2$  is to  $BD^2$ ; that is, every particular *lamina* of the cone  $ABC$  will be to a like *lamina* of the cone  $ABD$  as the base of the former cone is to the base of the latter; therefore *componendo*, the whole cone  $ABC$  will be to the whole cone  $ABD$  as the base of the former is to the base of the latter: but the planes  $ABC$  and  $ABD$  may be so constituted as to generate any two isosceles cones whatever that have equal heights; therefore if the heights of two isosceles cones be equal, these cones will be to each other as their bases. Q. E. D.

## A T H E O R E M.

358. Every isosceles pyramid is equal to an isosceles cone of an equal base and height.

Let  $P$  represent any isosceles pyramid, and  $C$  an isosceles cone of an equal base and height: I say then that the pyramid  $P$  will be equal to the cone  $C$ .

To demonstrate this, imagine the pyramid  $P$  to have a cone, as  $c$ , inscribed in it, so as to touch every side of the pyramid in so many lines of contact, and imagine the circumscribing pyramid, and consequently the inscribed cone, to be constituted of an infinite number of infinitely thin *laminæ*; and every *lamina* of the circumscribing pyramid will be to a correspondent *lamina* of the inscribed cone as the base of the pyramid is to the base of the cone. For the plane of every *lamina* that constitutes the pyramid will be a polygon similar to the base, and the plane of every correspondent *lamina* that constitutes the inscribed cone will be a circle inscribed in such a polygon: therefore *componendo*, all the *laminæ* constituting the pyramid  $P$  will be to all those that constitute the cone  $c$ , that is, the whole pyramid  $P$  will be to the whole cone  $c$  as the base of  $P$  is to the base of  $c$ : but the cone  $c$  is to the cone  $C$  of an equal height, as the base of  $c$  is to the base of  $C$ . Since then  $P$  is to  $c$  as the base of  $P$  is to the base of  $c$ , and  $c$  is to  $C$  as the base of  $c$  is to base of  $C$ , it follows *ex æquo* that  $P$  is to  $C$  as the base of  $P$  is to the base of  $C$ : but the base of  $P$  is equal to the base of  $C$  by the supposition; therefore the pyramid  $P$  is equal to the cone  $C$ , having an equal base and altitude. Q. E. D.

## C O R O L L A R Y.

Hence it follows, that whether cones be compared with cones, or cones with pyramids, or pyramids with pyramids, all such as have equal heights will be to one another as their bases. For cones are so by the last article, and pyramids are equal to cones having equal bases

bases and heights by this; I mean isosceles pyramids and isosceles cones.

### SCHOLIUM.

*As the solid content of a prism or a cylinder of a given perpendicular altitude depends upon the quantity, and not upon the figure of the base, so by the demonstration of this proposition it appears, that the solid content of an isosceles pyramid or cone of a given perpendicular altitude depends upon the quantity, and not upon the figure of the base: let the perpendicular altitude and the area of the base be the same, and the quantity of the solid will continue the same, whether that base be in the form of a triangle, or a square, or a polygon, or a circle. Other pyramids and cones will be considered in another place.*

### A L E M M A.

359. *If from the center of any cube straight lines be imagined to be drawn to all its angles, the cube will by this means be distinguished into as many equal isosceles pyramids as it has sides, to wit, 6, whose bases will be in the sides of a cube, and whose common vertex will be in the center.*

For if from a point above any right-lined plain figure lines be drawn to all its angles, and then the interstices of these lines be imagined to be filled up with triangular planes, the solid figure thus inclosed will be a pyramid; therefore, by the lines above described, the cube will be distinguished into as many pyramids as it hath sides. And that these pyramids will be all equal and isosceles, is evident: for first, their bases will be all equal from the nature of the cube; and in the next place, their heights will be all equal from the nature of the center, which is supposed to be equally distant from all the sides of the cube; and lastly, as this center must also be equally distant from all its angles, it follows that these pyramids must be all isosceles pyramids. Q. E. D.

COROL.

## COROLLARY.

Hence every one of these pyramids will be the sixth part of the whole cube.

## A THEOREM.

360. Every isosceles pyramid or cone is a third part of an isosceles prism or cylinder having an equal base, and an equal perpendicular height.

Let  $a$  be the perpendicular altitude of any isosceles pyramid or cone, and let  $b$  be the area of its base, both taken according to the directions given in art. 353: imagine also a cube whose side, that is the side of whose square base, is  $2a$ ; then will  $4a^2$  be the area of the base, and  $8a^3$  the solid content of this cube: and if, from the center of the cube, lines be imagined to be drawn to the four angles of the base, they will be the outlines of an isosceles pyramid whose base is the same with the base of the cube, to wit,  $4a^2$ , and whose perpendicular altitude is  $a$ ; whence the solid content of this pyramid will be  $\frac{8a^3}{6}$  or  $\frac{4a^3}{3}$ , by the corollary in the last article: but as this pyramid has the same height  $a$  with the pyramid proposed, these two pyramids will be to one another as their bases, by the corollary in the last article but one: hence the solid content of the pyramid proposed will easily be had by saying, as  $4a^2$ , the base of the pyramid within the cube, is to  $b$ , the base of the pyramid proposed, so is  $\frac{4a^3}{3}$  the solid content of the former, to a fourth, to wit,  $\frac{ab}{3}$ , which will be the solid content of the latter; therefore the solid content of an isosceles pyramid or cone whose base is  $b$ , and whose altitude is  $a$ , is found to be  $\frac{ab}{3}$ : but the solid content of an isosceles prism or cylinder having an

an equal base and height is  $ab$ ; therefore every isosceles pyramid or cone is a third part of an isosceles prism or cylinder having an equal base and an equal perpendicular altitude. *Q. E. D.*

COROLLARY 1.

*Hence the solid content of an isosceles pyramid or cone is found by multiplying the area of the base into a third part of the perpendicular altitude, or else by multiplying the area of the base into the whole altitude, and then taking a third part of the product.*

COROLLARY 2.

*Hence also it follows that all isosceles pyramids and cones upon equal bases are to one another as their heights.*

A L E M M A.

361. *If a pyramid of any kind be cut by a plane parallel to its base, the quantity of the section, or (which is all one) the quantity of the base of the pyramid cut off, will always be the same, let the figure of the pyramid be what it will, so long as the base and perpendicular altitude of the whole pyramid, and the perpendicular altitude of the pyramid cut off continue the same: in which case, the perpendicular distance of the plane of the section from the plane of the base will also be the same. (See Fig. 59.)*

Let  $A$  be the vertex of the pyramid, and let  $BC$  be any one side of the base; let the lines  $AB$  and  $AC$  be cut by the plane of the section in the points  $D$  and  $E$  respectively, and let  $AFG$  be the perpendicular altitude of the whole pyramid, cutting the plane of the section in  $F$ , and the plane of the base in  $G$ , both produced if need be: join  $FD$ ,  $FE$ ,  $GB$ ,  $GC$ : then since the base of the pyramid cut off will always be similar to the base of the whole pyramid, whereof  $DE$  and  $BC$  are correspondent sides; and since all similar plain figures are to each other as the squares of their correspondent

respondent sides by the 20th of the sixth book of the Elements, it follows that the base whose side is  $DE$  will be to the base whose side is  $BC$  as  $DE^2$  to  $BC^2$ , that is, by similar triangles, as  $AD^2$  is to  $AB^2$ , or  $AF^2$  is to  $AG^2$ . Since then as  $AG^2$  is to  $AF^2$ , so is the base of the whole pyramid to the base of the pyramid cut off; so long as the three first continue the same, the last must also continue the same. *Q. E. D.*

#### COROLLARY.

*Since the number of sides of the pyramid is not concerned in the demonstration of this proposition, which will be equally true, be the number of sides what it will, it must also be true of the cone, which is nothing else but a pyramid of an infinite number of sides, let the shape of the cone be what it will; that is, whether  $AG$  the perpendicular altitude of the cone passes through the center of the base or not.*

#### A THEOREM.

362. *If a prism or cylinder of any kind be described by the motion of a plain figure ascending uniformly in a horizontal position to any given height, the quantity of the solid thus generated will be the same, whether the describing plane descends directly or obliquely to the same height; and consequently all prisms and cylinders of what kind soever, that have equal bases and equal perpendicular heights, are equal, whether they stand upon those bases erect or reclining.*

For the better conceiving of this, let the describing plane be made, not to ascend all the way, but sometimes to ascend perpendicularly, and sometimes to move laterally or edgeway, and that by turns: then it is plain that the quantity of solid space, or rather the sum of all the solid spaces thus described, will amount to no more than if the describing plane had ascended all the way perpendicularly to the same height. Let the times of these alternate motions wherein

wherein they are performed be diminished and their number be increased *ad infinitum*, and they will terminate at last in an uniform oblique motion, and the solid generated by this motion will be equal to a solid generated by a perpendicular motion of the same plane to the same height. *Q. E. D.*

*N. B.* What has here been demonstrated concerning prisms and cylinders, may be further illustrated by sliding a pack of cards, or a pile of halfpence, out of an erect into an oblique posture; whereby it may easily be seen, that neither the base or the perpendicular altitude, nor the quantity of the solid, can be affected by this change of posture; but the finer, that is the thinner, these constituent *laminae* are, the nearer they will represent an oblique solid.

#### A T H E O R E M.

363. *All pyramids and cones of what kind soever, that have equal bases and equal perpendicular heights, are equal.*

To evince this, let us imagine two plain figures (whether similar or dissimilar to each other it matters not) to rise together from the same level, one directly, and the other obliquely, but both in a horizontal position, and always upon the same level; and let these planes be imagined not to retain all along their first magnitude (as was supposed in the last article) but to lessen by degrees as they rise, so as by their motion to describe tapering figures, and at last to vanish each in a point: then it is easy to see, that if the tapering figures thus described be pyramids or cones having equal bases and equal perpendicular heights, these describing planes must not only be equal to each other at first, and vanish at equal heights, but they must lessen so together as to be equal to each other at all other equal altitudes whatever: this is evident from the last article but one: and therefore the solids described by them must necessarily be equal. *Q. E. D.*

## COROLLARY.

Hence it follows, that whatever we have demonstrated concerning isosceles pyramids, cones, prisms, and cylinders, with respect to their proportion one to another, will be equally true of all others, whatever shape or posture they may be in: as, that all pyramids and cones of the same height are to each other as their bases, that all pyramids and cones upon equal bases are as their heights, and that every pyramid or cone is a third part of a prism or cylinder having an equal base, and an equal perpendicular altitude.

## A L E M M A. (Fig. 60.)

364. Let  $ABCD$  be a square whose base is  $AD$ , and whose diagonal is  $AC$ ; and upon the center  $A$ , and with the radius  $AB$ , describe the quadrant or quarter of a circle  $BAD$ : draw also the line  $EFGH$  or  $EGFH$  any where within the square, parallel to the base  $AD$ , cutting the side  $AB$  in  $E$ , the quadrant  $BD$  in  $F$ , the diagonal  $AC$  in  $G$ , and the opposite side  $CD$  in  $H$ , and join  $AF$ : I say then that the square of  $EF$  and the square of  $EG$  put together will always be equal to the square of  $EH$ .

For the triangles  $ABC$  and  $AEG$  are similar, as having one angle at  $A$  in common, and the angles at  $B$  and  $E$  right; therefore  $EG$  will be to  $EA$  as  $BC$  is to  $BA$ ; but  $BC$  is equal to  $BA$ , from the nature of a square; therefore  $EG$  will be equal to  $EA$ , and  $EG^2$  to  $EA^2$ , and  $EF^2 + EG^2$  to  $EF^2 + EA^2 = AF^2 = AD^2 = EH^2$ , that is,  $EF^2 + EG^2 = EH^2$ . Q. E. D.

## A T H E O R E M.

365. Every sphere is two-thirds of a circumscribing cylinder, that is, a cylinder that will just contain it.

For supposing all things as in the last article, (see Fig. 60.) let the square  $ABCD$  be now supposed

to turn round its fixed side  $AB$ , so that the square may generate a cylinder, the quadrant a hemisphere, and the triangle  $ABC$  an inverted cone; and let this cylinder, and consequently the cone and hemisphere, be considered as consisting of an infinite number of infinitely thin cylindric *laminæ*: then if  $EH$  represents the semidiameter of any one of these *laminæ* belonging to the cylinder,  $EG$  will be the semidiameter of so much of this *lamina* as lies within the cone, and  $EF$  the semidiameter of so much as lies within the hemisphere: and because (by the last article) the square of  $EF$  and the square of  $EG$  are both together equal to the square of  $EH$ , it follows from corollary 6 in art. 347, that the two circles, whose semidiameters are  $EF$  and  $EG$ , are both together equal to the circle whose semidiameter is  $EH$ ; which is as much as to say in other words, since the line  $EH$  was taken indifferently, that every particular *lamina* of the cylinder is equal to two correspondent *laminæ* at the same height, whereof one belongs to the cone, and the other to the hemisphere; therefore, *componendo*, the whole cylinder is equal to the cone and the hemisphere put together: but the cone has been proved already to be a third part of the cylinder, as having the same base and height, (see art. 360); therefore the hemisphere must be the remaining two-thirds of it; that is, every hemisphere is two-thirds of a cylinder of the same base and height.

Let us now imagine two such hemispheres, and two such cylinders to be put together in one common base, and the two hemispheres will constitute a sphere, and the two cylinders a cylinder circumscribed about that sphere, and the sphere will now be two thirds of the circumscribing cylinder.  
 Q. E. D.

## A T H E O R E M.

366. *Every sphere is equal to a cone or pyramid whose base is the surface of the sphere, and whose perpendicular altitude is its semidiameter.*

To demonstrate this, let a body terminated by plain sides, regular or irregular, be so constituted as to admit of a sphere to be inscribed in it, touching every side in some point or other, as the cube and an infinite number of other bodies will; and from the center of the inscribed sphere imagine lines to be drawn to every angle of the circumscribing body: then will this body be distinguished into as many pyramids as it hath sides, whose bases will be the several sides of the body, whose common vertex will be in the center of the sphere, and whose perpendicular heights will be *radii* drawn to the several points of contact, and consequently will be equal: for as when a circle is touched by right lines, all *radii* drawn to the several points of contact will be perpendicular to the respective tangent lines; so when a sphere is touched by planes, all *radii* drawn to the several points of contact will be perpendicular to the respective tangent planes.

Let then  $r$  be the *radius* of the sphere, and let  $a, b, c, d$ , represent the quantities or areas of the several sides of the circumscribing body; and the solid contents of the pyramids whereof that body is composed will be  $\frac{ar}{3}, \frac{br}{3}, \frac{cr}{3}, \frac{dr}{3}$ , and the sum of all

these pyramids, or the solid content of the body, will be  $\frac{ar+br+cr+dr}{3}$ . Let  $a+b+c+d=s$ ,

that is, let  $s$  be the whole surface of the circumscribing body, and its solid content will be  $\frac{rs}{3}$ : but  $\frac{rs}{3}$  is the content of a pyramid whose base is  $s$ , and whose perpendicular altitude is  $r$ ; therefore every body  
circum-

circumscribed about a sphere is equal to a pyramid whose base is the surface of the body, and whose perpendicular altitude is the semidiameter of the inscribed sphere.

Let us now imagine the several angles of this circumscribing body to be pared off close by the surface of the sphere, so as to create more sides and more angles, and we shall still have a body circumscribed about the sphere, though in another shape; and therefore the proportion already advanced will be as true in relation to this body as to the former: whence it follows, that if we suppose the angles of the circumscribing body to be pared off *ad infinitum*, that is, till it differs nothing from the inscribed sphere, the proposition will be as true of the sphere itself as it was before of the body circumscribed about it, to wit, that every sphere is equal to a cone or pyramid whose base is the surface of the sphere, and whose perpendicular altitude is its semidiameter. *Q. E. D.*

### A T H E O R E M.

367. *The surface of every sphere is equal to four great circles of the same sphere.*

Where note, that by a great circle of a sphere, I mean any circle arising from a section of a sphere into two equal hemispheres by a plane passing through its center, in contradistinction to a lesser circle arising from a section of a sphere into unequal parts: or a great circle of a sphere may be defined to be a circle whose diameter is the same with that of the sphere.

Let  $s$  be the surface of any sphere,  $d$  the diameter, and  $b$  the area, of a great circle of that sphere; then will  $b$  be the base of a circumscribing cylinder,  $d$  its altitude, and  $bd$  its solid content; therefore, by the last article but one,  $\frac{2bd}{3}$  will be the solid content of the sphere: but by the last article, this sphere is equal to a cone or pyramid whose base is  $s$ , the surface

of the sphere, and whose perpendicular altitude is  $\frac{d}{2}$  its semidiameter, a third part whereof is  $\frac{d}{6}$ ; there-

fore  $\frac{s d}{6}$  will also represent the solid content of the sphere: whence we have the following equation, to wit,  $\frac{s d}{6} = \frac{2 b d}{3}$ , which being reduced gives  $s = 4 b$ .

Q. E. D.

From the three last articles may be deduced the following corollaries:

### COROLLARIES.

368. 1st, *As the diameter of a circle is to the circumference, that is, as 7 to 22 nearly, so is the square of the diameter of any sphere to its surface.* For supposing the diameter of a circle to be to the circumference as 1 to  $c$ , and putting  $d$  for the diameter of any sphere,  $cd$  will be the circumference of a great circle of that sphere, since as 1 is to  $c$ , so is  $d$  to  $cd$ ; multiply then  $\frac{cd}{2}$  the semicircumference, into  $\frac{d}{2}$  the

radius, and you will have  $\frac{cdd}{4}$  the area of a great circle; therefore four great circles, or the surface of the sphere, will be  $cdd$ : but as 1 is to  $c$ , so is  $dd$  to  $cdd$ ; therefore, &c.

2d, *Whence it follows, that The surface of every sphere is equal to the product of the circumference of a great circle multiplied into the diameter of the sphere.* For, retaining the notation of the last article,  $cdd$  the surface of the sphere is equal to  $cd$  the circumference of a great circle multiplied into  $d$  the diameter.

3d, *The surface of every sphere is equal to the convex surface of a circumscribed cylinder.* For if a concave cylinder without its two bases be slit, and then opened into a plane, the figure of that plane will be a parallelogram,

lelogram, whose base will be that line which before was the circumference of the base of the cylinder, and whose height will be the same with that of the cylinder; therefore, as the area of a parallelogram is found by multiplying the base into the height, the surface of every cylinder must be found by multiplying the circumference of the base into the height of the cylinder: but the circumference of a cylinder circumscribed about a sphere is equal to the circumference of a great circle of the sphere, and the height of such a cylinder is equal to the diameter of the sphere; therefore the convex surface of the cylinder will be equal to the circumference of a great circle of the sphere multiplied into the diameter, which by the last corollary is the surface of the inscribed sphere.

4th, *The solid content of every sphere is equal to the product of its surface multiplied into a third part of the radius, or the radius into a third part of the surface.* This is evident from art. 366.

5th, *As six times the diameter of a circle is to the circumference, that is, as 42 is to 22 or 21 to 11 nearly, so is the cube of the diameter of any sphere to its solid content.* For if we suppose the diameter of a circle to be to the circumference as 1 to  $c$ , the surface of a sphere whose diameter is  $d$  will be  $cdd$  by the first corollary; and this surface multiplied into a third part of the radius, or into a third part of  $\frac{d}{2}$ , which is

$\frac{d}{6}$  gives  $\frac{cd^3}{6}$  the solid content of the sphere: but as

6 is to  $c$ , so is  $d^3$  to  $\frac{cd^3}{6}$ ; therefore as six times the diameter of a circle is to the circumference so is the cube of the diameter of any sphere to its solid content.

6th, *The surfaces of all spheres are as the squares, and the solid contents as the cubes, of their diameters or semidiameters.* For supposing the diameter of any

circle to be to the circumference as 1 to  $c$ , and supposing  $d$  and  $e$  to be the diameters of two spheres, the surfaces will be  $cd^2$  and  $ce^2$  by the first corollary, and the solid contents will be  $\frac{cd^3}{6}$  and  $\frac{ce^3}{6}$  by the last: but  $cd^2$  is to  $ce^2$  as  $d^2$  is to  $e^2$ , or as  $\frac{d^2}{4}$  is to  $\frac{e^2}{4}$ ; and  $\frac{cd^3}{6}$  is to  $\frac{ce^3}{6}$  as  $d^3$  is to  $e^3$ , or as  $\frac{d^3}{8}$  is to  $\frac{e^3}{8}$ .

To shew the use of the properties of the sphere above described, I shall add the following problems:

### PROBLEM I.

369. *To find how many acres the surface of the whole earth contains.*

Let the diameter of a circle be to the circumference as  $d$  to  $c$ , and let  $e$  be the circumference of the earth; then will  $\frac{de}{c}$  be its diameter, and  $\frac{de^2}{c}$  its surface by the second corollary in the last article. Now the circumference of the earth is 131630573 English feet, or 24930 English miles nearly, allowing 5280 feet to a mile: therefore if we make  $e=24930$ , we shall have  $e^2=621504900$ . Now the numbers 7 and 22 are scarcely exact enough to express the proportion of the diameter of a circle to the circumference in company with so large a number as  $e^2$ ; let us therefore use that of 113 to 355, which we have elsewhere shewn (schol. 1. in art. 179\*) to be much more exact; that is, let  $d=113$  and  $c=355$ , and  $\frac{de^2}{c}$  or the surface of the earth will be 197831137 square miles: but every square mile contains 640 acres; therefore, if the foregoing number of square miles be multiplied by 640, the product 126611927680 will be the number of acres required.

\* See the Quarto Edition, p. 282.

*N. B.* To be more exact in any computation than the *data* on which it is founded, can be to little or no purpose.

P R O B L E M 2.

370. *What must be the diameter of a concave sphere that will just hold an English gallon?*

By the fifth corollary in art. 368, as 11 is to 21 \*, so is the solid content of any sphere to the cube of its diameter : but the solid content of our sphere is 282 cubic inches, or an English gallon, by the supposition : therefore the cube of its diameter will be  $538\frac{4}{11}$ , the cube root whereof 8.135 will be the diameter itself.

*N. B.* The extraction of the cube root is taught in most books of Arithmetic, and depends on the nature of a binomial, as doth the extraction of the square root ; and therefore whoever sees the reason of the latter, may (without much difficulty) reason himself into the former : but the extraction of the roots of all simple powers will best be performed by the help of logarithms, as will be shewn hereafter when we come to treat of the nature and properties of those numbers.

*Of the Spheroid.*

373. *If a sphere be resolved into an infinite number of infinitely thin cylindric laminæ, and then these laminæ, retaining their circular figure, be all increased or all diminished in the same proportion, they will constitute a figure called a spheroid; and it is said to be prolate or oblong, according as these constituent laminæ are increased or diminished.* This a learner, who is unacquainted with the nature of the *ellipses*, may (if he pleases) take for the definition of a spheroid.

From the definition here given it follows,

1st, that *Every spheroid is to a sphere upon the same axis, as any one lamina in the former is to a like lamina in the latter from whence it was derived; or as any*

\* Or as 4 to 2 —  $\frac{1}{12}$ .

number of laminæ in the former is to the same number of like laminæ in the latter, that is, as any portion of the former comprehended between two parallel planes perpendicular to its axis, is to a like portion of the latter.

2dly, it follows, that Every spheroid, as well as every sphere, is two thirds of a circumscribing cylinder. For though a spheroid be greater or less than a sphere upon the same axis, the cylinder circumscribed about the spheroid will be proportionably greater or less than the cylinder circumscribed about the sphere: for, having the same length, they will be as their bases; therefore the spheroid will have the same proportion to a cylinder circumscribed about it, as the sphere hath to a cylinder circumscribed about the sphere.

### A L E M M A.

374. *The chord of any circular arc is a mean proportional between the versed sine of that arc and the diameter.*

Let  $ABC$  (Fig. 61.) be a semicircle whose diameter is  $AC$ , and assuming any arc as  $AB$ , draw the straight line  $AB$ , which is its chord; draw also  $BD$  perpendicular to the diameter  $AC$  in  $D$ , and the intercepted line  $AD$  is called the versed sine of the arc  $AB$ . What we are then to demonstrate is, that the chord  $AB$  is a mean proportional between the versed sine  $AD$  and the whole diameter  $AC$ : and this is easily done by drawing the other chord  $BC$ ; for then the triangle  $ABC$  will be right-angled at  $B$ , as being in a semicircle, and consequently will be similar to the right-angled triangle  $ADB$ ; whence  $AD$  will be to  $AB$  as  $AB$  to  $AC$ . Q. E. D.

### P R O B L E M 5.

375. *To find the solid content of a frustum of a hemisphere or hemispheroid comprehended between a great circle perpendicular to its axis and any other lesser circle parallel to it, having these two opposite bases and the height of the frustum given.*

N. B.

N. B. As  $\square AD$  is sometimes used for the square of  $AD$ , or a square whose side is  $AD$ , so in our notation in this and some of the following articles, we shall not scruple to use  $\odot AD$  for the area of a circle whose semidiameter is  $AD$ ,  $2 \odot AD$  for two such circles, &c.

Let  $ABCD$  (Fig. 60.) be a square whose base is  $AD$  and diagonal  $AC$ ; and upon the center  $A$  and with the radius  $AB$  describe the quadrant  $BAD$ ; draw also the line  $EFGH$  any where within the square parallel to  $AD$ , cutting  $AB$  in  $E$ , the quadrant in  $F$ , the diagonal in  $G$ , and the opposite side  $CD$  in  $H$ . This done, imagine the whole figure to turn round its fixed side  $AB$ : then will the square generate a cylinder, the quadrant a hemisphere, the triangle  $ABC$  an inverted cone, and the curvilinear space  $AEFD$  such a frustum of an hemisphere as we are to find the solid content of, having given  $AD$  and  $EF$  the semidiameters of the two opposite bases, and  $AE$  the height of the frustum.

In the 365th article, by the help of this construction, it was demonstrated, that the hemisphere generated by the quadrant  $ABD$  and the cone generated by the triangle  $ABC$  were together equal to the cylinder generated by the square  $ABCD$ ; and the reasons there given for such an equality, equally prove that the frustum generated by the space  $AEFD$  and the cone generated by the triangle  $AEG$  will both together be equal to the cylinder generated by the parallelogram  $AEHD$ : but the cone generated by the triangle  $AEG$  is equal to  $\odot EG \times \frac{AE}{3}$ ; and the cylinder generated by the parallelogram  $AEHD$  is equal to  $\odot AD \times AE = 3 \odot AD \times \frac{AE}{3} = 2 \odot AD + \odot EH \times \frac{AE}{3}$ , therefore, if  $f$  be put for the solid content of the frustum, we shall have the following equation,

$$3 \qquad f +$$

$$f + \odot EG \times \frac{AE}{3} = \overline{2 \odot AD + \odot EH} \times \frac{AE}{3}; \text{ trans-}$$

pose  $\odot EG \times \frac{AE}{3}$ , and then we shall have  $f =$

$$\overline{2 \odot AD + \odot EH - \odot EG} \times \frac{AE}{3} : \text{ but by the 364th ar-}$$

ticle, and the sixth corollary in the 347th,  $\odot EH = \odot EF + \odot EG$ ; therefore  $\odot EH - \odot EG = \odot EF$  : substitute  $\odot EF$  instead of  $\odot EH - \odot EG$  in the fore-

$$\text{going equation } (f = \overline{2 \odot AD + \odot EH - \odot EG} \times \frac{AE}{3})$$

and you will have  $f = \overline{2 \odot AD + \odot EF} \times \frac{AE}{3}$  : this is

upon a supposition that the solid proposed is a frustum of a hemisphere. Let us now suppose the solid  $f$  to consist of an infinite number of infinitely thin cylindric *laminae* parallel to its base, and then that these *laminae*, retaining their circular figure, be all diminished in some given proportion, suppose in the proportion of  $r$  to  $s$ ; then it is plain that the solid  $f$  will degenerate into a frustum of an hemispheroid, and that it will be diminished in the proportion of  $r$  to  $s$ ;

but then the quantity  $\overline{2 \odot AD + \odot EF} \times \frac{AE}{3}$  will al-

so be diminished in the same proportion; and therefore

$$f \text{ will still be equal to } \overline{2 \odot AD + \odot EF} \times \frac{AE}{3}; \text{ whence}$$

we have the following theorem for finding the solid content of the frustum proposed, whether it be a frustum of a hemisphere or hemispheroid.

*To twice the area of the greater base add the area of the less; multiply the sum by a third part of the altitude of the frustum, and the product will be its solid content.*  
Q. E. I.

## P R O B L E M 6.

376. *To find the convex surface of any segment of a sphere whose base and height are given.* (Fig. 60.)

Retaining the construction of the last article, and supposing what was there proved, if from the hemisphere generated by the space  $ABD$  be subtracted the frustum generated by the space  $AEFD$ , there will remain a segment of the sphere generated by the space  $BEF$ ; and if to this segment again be added the cone generated by the triangle  $AEF$ , they will both together constitute a sector of the sphere generated by the space  $ABF$ ; and lastly, if the solid content of this spherical sector be applied to, or divided by a third part of the radius  $AD$ , the plane or quotient thence arising will be equal to the convex surface generated by the arc  $BF$ , which is here proposed to be determined. For as every sphere is equal to a cone whose base is its surface and whose altitude is its radius, (see art. 366) so (and for the same reason) must every sector of a sphere be equal to a cone whose base is the spherical part of its surface, and whose altitude is the radius. Now the hemisphere generated by the space  $ABD$  being two-thirds of a cylinder of the same base and height, as was demonstrated in art. 365, its solid content will be expressed by  $2 \odot AD \times \frac{AB}{3} = 2 \odot AD \times \frac{AE}{3} + 2 \odot AD \times \frac{EB}{3}$ ; and the solid content of the frustum generated by the space  $AEFD$  was  $2 \odot AD \times \frac{AE}{3} + \odot EF \times \frac{AE}{3}$ ; subtract the latter from the former, and there will remain the segment generated by the space  $BEF$  equal to  $2 \odot AD \times \frac{EB}{3} - \odot EF \times \frac{AE}{3}$ ; add to this the cone generated by the triangle  $AEF$ , whose content is  $\odot EF \times \frac{AE}{3}$ , and you will have the spherical sector generated by the

space  $ABF$  equal to  $2 \odot AD \times \frac{EB}{3}$ . Let the diameter of a circle be to the circumference as 1 to  $c$ , and  $2AD \times c$  will be the circumference of a great circle, whose half  $AD \times c$  multiplied into  $AD$  the *radius*, will give  $AD^2 \times c$  for the area of a great circle; therefore  $\odot AD = AD^2 \times c$ , and  $2 \odot AD \times \frac{EB}{3}$ , or the content of the sector, will be  $2AD^2 \times c \times \frac{EB}{3}$ : but  $EB$  is the versed sine of the arc  $BF$ ; and therefore if we put  $l$  for the chord of that arc, we shall have  $2AD \times EB = l^2$  by the last article but one; and the solid content of the sector will now be  $l^2 \times c \times \frac{AD}{3}$ ; divide by  $\frac{AD}{3}$ , and you will have the surface generated by the arc  $BF$  equal to  $l^2 \times c$ : but as  $AD^2 \times c$  was equal to  $\odot AD$ , so will  $l^2 \times c$  be equal to  $\odot l$ , that is, to a circle whose *radius* is the chord of the arc  $BF$ : therefore *the surface of every segment of a sphere is equal to a circle whose radius is the distance of the pole, or vertical point of the segment, from the circumference of its base.*

What has here been determined concerning the convex surface of a segment of a sphere agrees entirely with what was determined in art. 367, concerning the surface of a whole sphere. For if we suppose the arc  $BF$  to be a semicircle, its chord will then be a diameter, and the surface generated by this arc will be the surface of the whole sphere; and therefore the surface of this sphere will be equal to a circle whose *radius* is the diameter of the sphere, that is  $2AD$ : but a circle whose *radius* is  $2AD$ , is quadruple of a circle whose *radius* is  $AD$ , because all circles are as the squares of their semidiameters; therefore the surface of every sphere is equal to four great circles of the same, as was there demonstrated.

*N. P.* They who would see more of mensuration are referred to an excellent treatise on that subject by Dr. *Hutton*.

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THE  
ELEMENTS OF ALGEBRA.  

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BOOK IX. PART I.

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*Of Powers and their Indexes.*

378. **T**HE indexes of powers have been already considered, so far as they serve for a sort of short-hand writing in Algebra: but the incomparable *Newton* has very much enlarged our views with respect to these indexes or exponents, in so much that it is by their means chiefly, that so many excellent, useful, and comprehensive theorems have been discovered both in Algebra and Geometry, and more particularly in the doctrine of Fluxions. This sort of notation therefore I shall now endeavour further to explain in my observations upon the following small table:

Powers without their Indexes.

xxxxx.	xxxx.	xxx.	xx.	x.	1.	$\frac{1}{x}$ .	$\frac{1}{xx}$ .	$\frac{1}{xxx}$ .
						$\frac{1}{xxxx}$ .	$\frac{1}{xxxxx}$ .	

Powers with their Indexes.

$x^5$ .	$x^4$ .	$x^3$ .	$x^2$ .	$x^1$ .	$x^0$ .	$x^{-1}$ .	$x^{-2}$ .	$x^{-3}$ .
				$x^{-4}$ .	$x^{-5}$ .			

This table consists of two rows, whereof the upper is a series of powers expressed without their indexes, the common root or fundamental quantity being  $x$ ; the lower expresses the same powers by the help of their indexes.

OBSER.

## OBSERVATIONS.

379. 1st, *By this table it appears that every subsequent power is the quotient of the next before it divided by the common root  $x$ , and that every subsequent index is generated by subtracting unity from the next before it.* Thus  $x^2$  divided by  $x$  gives  $x$ ,  $x$  divided by  $x$  gives 1,

1 divided by  $x$  gives  $\frac{1}{x}$ ,  $\frac{1}{x}$  divided by  $x$  gives  $\frac{1}{xx}$ ,

&c.: thus again,  $2 - 1 = 1$ ,  $1 - 1 = 0$ ,  $0 - 1 = -1$ ,  $-1 - 1 = -2$ , &c. Since then each row exhibits a regular series, it follows that the negative indexes have the same right to express the powers they belong to as the affirmative ones, and that  $x^{-2}$  represents

$\frac{1}{xx}$  upon the same foundation that  $x^2$  represents  $xx$ .

2dly, *Therefore whatever number is the index of any power, its negative will be the index of the reciprocal of that power, or of unity divided by that power.* Thus if 2 be the index of  $xx$ ,  $-2$  will be the index of

$\frac{1}{xx}$ ; if 1 be the index of  $x$ ,  $-1$  will be the index of

$\frac{1}{x}$ ; and so of the rest.

3dly, *In all cases whatever, the addition of indexes answers to the multiplication of the powers to which they belong; that is, if any two powers of the same quantity be multiplied together, the index of the multiplicator added to the index of the multiplicand will give the index of the product.* Thus  $x^2$  multiplied into  $x^3$  gives  $x^5$ , as  $xx \times xxx$

gives  $xxxxx$ : thus  $x^2 \times x^{-3}$  gives  $x^{-1}$ , as  $xx \times \frac{1}{xxx}$

gives  $\frac{1}{x}$ : thus  $x^{-2} \times x^{-3}$  gives  $x^{-5}$ , as  $\frac{1}{xx} \times \frac{1}{xxx}$  gives

$\frac{1}{xxxxx}$ : thus  $x^2 \times x^{-2}$  gives  $x^0$ , as  $xx \times \frac{1}{xx}$  gives 1:

$x^3 \times x^0$  gives  $x^3$ , as  $xxx \times 1$  gives  $xxx$ .

4thly,

4thly, *In like manner the subtraction of indexes answers to the division of powers; that is, if any power of any quantity be divided by a power of the same quantity, the index of the divisor subtracted from the index of the dividend leaves the index of the quotient.* Thus  $x^3$  divided by  $x^2$  quotes  $x^1$ , as  $xxx$  divided by  $xx$  quotes  $x$ : thus  $x^2$  divided by  $x^{-3}$  quotes  $x^5$ , as  $xx$  divided by  $\frac{1}{xxx}$  quotes  $xxxxx$ : thus  $x^{-2}$  divided by  $x^3$  quotes  $x^{-5}$ , as  $\frac{1}{xx}$  divided by  $xxx$  quotes  $\frac{1}{xxxxx}$ : thus  $x^{-2}$  divided by  $x^{-3}$  gives  $x^1$ , as  $\frac{1}{xx}$  divided by  $\frac{1}{xxx}$  gives  $x$ : thus  $x^0$  divided by  $x^{-1}$  gives  $x^1$ , as  $1$  divided by  $\frac{1}{xx}$  gives  $xx$ : lastly,  $x^2$  divided by  $x^2$  gives  $x^0$ , as  $xx$  divided by  $xx$  gives  $1$ .

5thly, *If the index of any power be multiplied by 2, 3, 4, &c. the product will be the index of the square, cube, square square, &c. of that power: and therefore if the index of any power be divided by 2, 3, 4, &c. the quotient will be the index of the square root, cube root, square-square root, &c. of that power.* Thus the square of  $x^2$  is  $x^4$ , its cube  $x^6$ , its square-square  $x^8$ : thus again, the square root of  $x^{12}$  is  $x^6$ , its cube root  $x^4$ : its square-square root  $x^3$ , &c.: thus the square root of  $x$  or  $x^1$  is  $x^{\frac{1}{2}}$ , its cube root  $x^{\frac{1}{3}}$ , its square-square root  $x^{\frac{1}{4}}$ , &c.: thus the square root of  $\frac{1}{x}$  or  $x^{-1}$  is  $x^{-\frac{1}{2}}$ , its cube root  $x^{-\frac{1}{3}}$ , its square-square root  $x^{-\frac{1}{4}}$ , &c. thus  $x^{\frac{2}{3}}$  signifies the cube root of  $x^2$ ,  $x^{\frac{3}{4}}$  the square-square root of  $x^3$ . And universally,  $x^{\frac{m}{n}}$  signifies that root of  $x^m$  whose index is  $n$ ; as if  $y^n = x^m$ , then  $y$  is said to be that root of  $x^m$  whose

index is  $n$ , and must be expressed by  $x^{\frac{m}{n}}$ ; and therefore if in any case  $x^m = y^n$ , it will be a good inference to say that  $y$  is equal to  $x^{\frac{m}{n}}$ , or that  $x$  is equal to  $y^{\frac{n}{m}}$ .

6thly, Powers are reducible to more simple powers, as often as their fractional indexes are reducible to more simple fractions. Thus the square-square root of  $x^2$  is the same with the square root of  $x$ , because  $x^{\frac{2}{4}} = x^{\frac{1}{2}}$ .

7thly, If the index of any power be an improper fraction, and that fraction be reduced into a whole number and a fraction, the power will hereby be resolved into two factors, whereof one will have the whole number for its index, and the other the fractional part.

Thus  $\frac{5}{2} = 2 + \frac{1}{2}$ , and therefore  $x^{\frac{5}{2}} = x^2 \times x^{\frac{1}{2}}$ ; that is, the square root of  $x^5$  is equal to  $xx$  multiplied into the square root of  $x$ .

8thly, Surd powers may be reduced to the same root by a reduction of their fractional indexes to the same denomination, and that, whether they be powers of the same quantity or not. Thus  $x^{\frac{1}{2}}$  and  $y^{\frac{1}{3}}$  are the same as  $x^{\frac{3}{6}}$  and  $y^{\frac{2}{6}}$ ; that is, the square root of  $x$  and the cube root of  $y$  are the same as the sixth root of  $x^3$  and the sixth root of  $y^2$ , and thus may surds of different roots be compared together without any extraction of those roots. As for instance, if any one should ask me, which of these two quantities is the greater, the square root of 2, or the cube root of 3? I should answer, the cube root of 3; for the square root of 2 or  $2^{\frac{1}{2}}$  or  $2^{\frac{3}{6}}$  is equal to  $8^{\frac{1}{6}}$ ; but the cube root of 3, or  $3^{\frac{1}{3}}$ , or  $3^{\frac{2}{6}}$  is equal to  $9^{\frac{1}{6}}$ ; and  $9^{\frac{1}{6}}$  is greater than  $8^{\frac{1}{6}}$ .

9thly,

9thly, *That the addition and subtraction of indexes answers to the multiplication and division of the powers to which they belong, holds equally true in fractional indexes, as in integral ones.* Thus  $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ , and  $x^{\frac{1}{2}} \times x^{\frac{1}{3}} = x^{\frac{5}{6}}$ , which I thus demonstrate. Let  $y^6 = x$ ; then by the fifth observation we shall have  $y = x^{\frac{1}{6}}$ ,  $y^3 = x^{\frac{3}{6}}$ , or  $x^{\frac{1}{2}}$ ,  $y^2 = x^{\frac{2}{6}}$ , or  $x^{\frac{1}{3}}$ , and  $y^5 = x^{\frac{5}{6}}$ : but  $y^3 \times y^2$  is equal to  $y^5$  by the third observation; therefore  $x^{\frac{1}{2}}$  multiplied into  $x^{\frac{1}{3}}$  gives  $x^{\frac{5}{6}}$ . After the same manner, since  $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ , it may be demonstrated that  $x^{\frac{1}{2}}$  divided by  $x^{\frac{1}{3}}$  will give  $x^{\frac{1}{6}}$ ; for  $y^3$  divided by  $y^2$  gives  $y$ , which is equal to  $x^{\frac{1}{6}}$ ; and the demonstrations will be the same in all other cases.

## P A R T II.

### *Of Logarithms, and their Uses.*

*The Definition of Logarithms, and Consequences drawn from it.*

Art. 390. **L**OGARITHMS are a set of artificial numbers placed over-against the natural ones, usually from 1 to 100000, and so contrived that their addition answers to the multiplication of the natural numbers to which they belong; that is, if any two numbers be multiplied together, and so produce a third, their logarithms being added together will constitute the logarithm of that third.

Thus 0.3010300, the common logarithm of 2, added to

0.4771213, the logarithm of 3, gives

0.7781513, the logarithm of 6, because 6 is the product of 2 and 3 multiplied together.

From this definition it follows, first, *That in any system or table of logarithms whatever, the logarithm of unity or 1 will be nothing* : for as 1 neither increases nor diminishes the number multiplied by it, so neither will its logarithm either increase or diminish the logarithm to which it is added ; and therefore the logarithm of 1 must be nothing.

2dly, *For a like reason, the logarithm of a proper fraction will always be negative* : for such a fraction always diminishes the number multiplied by it, and therefore its logarithm will always diminish the logarithm to which it is added.

3dly, *This property of logarithms, whereby they are defined as above, affords us no small compendium in multiplication* : for whenever one number is to be multiplied by another, it is but taking out their logarithms, and adding them together, and their sum will be a third logarithm whose natural number being taken out of the tables will be the product required.

4thly, *The subtraction of logarithms answers to the division of the natural numbers to which they belong* ; that is, whenever one number is to be divided by another, it is but subtracting the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient : and thus by the help of logarithms may the operation of division be performed by mere subtraction as that of multiplication was by addition. Hence as every fraction is nothing else but the quotient of the numerator divided by the denominator, its logarithm will be found by subtracting the logarithm of the denominator from the logarithm of the numerator. To demonstrate this, to wit, that the logarithm of the divisor subtracted from the logarithm of the dividend will leave the logarithm of the quotient, let the number *A* be divided by the number *B*, and let the quotient be the number *C*; and let the logarithms of the numbers *A*, *B*, and *C*, be *a*, *b*, and *c* respectively ; I say then that  $a - b$  will be equal to  $c$  : for since by the supposition

sition  $\frac{A}{B} = C$ , we shall have  $A = BC$ ,  $a = b + c$  by the definition; whence  $a - b = c$ .

5thly, *As every fourth proportional is found by multiplying the second and third numbers together, and dividing the product by the first, so the logarithm of every such fourth proportional will be found by adding the logarithms of the second and third numbers together, and subtracting from the sum the logarithm of the first.* This renders all operations by the rule of proportion very compendious and easy; especially after the practitioner has pretty well inured himself to take out of the table logarithms to his numbers, and numbers to his logarithms: but this compendium is chiefly useful in Trigonometry, both plain and spherical, where every thing he wants is put down ready to his hands.

6thly, *If A be any number whose logarithm is a, then the logarithm of  $A^2$  will be 2a, that of  $A^3$ , 3a, &c. that of  $\frac{1}{A}$ , -a, that of  $\frac{1}{A^2}$ , -2a, &c. And universally, the logarithm of  $A^m$  will be  $a \times m$ , and that, whether the index m be integral or fractional, affirmative or negative: on the other hand, if q be the logarithm of any power of A, as of  $A^m$ , then  $\frac{q}{m}$  will be the logarithm of A.* The reason of all this is plain; for as  $A^2$  is the product of A multiplied into itself, so its logarithm will be the logarithm of A added to itself or doubled, that is 2a; and so of the higher powers. Again, as  $\frac{1}{A}$  is the quotient of unity divided by A, its logarithm will be found by subtracting a, the logarithm of A, from 0, the logarithm of 1, which gives -a; and so of the lower powers. Lastly, as  $\sqrt{A}$ , when multiplied into itself, produces A, so its logarithm, when added to itself, ought to make a; therefore the logarithm of  $\sqrt{A}$  will be  $\frac{1}{2}a$ ; and so of all the other fractional powers. Here then again

we have another instance of the very great usefulness of a good table of logarithms, to wit, in raising a number to any given power, or in extracting any given root out of it, all which is performed with equal facility, only by multiplying its logarithm by the index of the given power, or dividing it by the index of the given root; as doubling it for the square, tripling it for the cube, &c.; halving it for the square root, trisecting it for the cube root, &c.: this, I say, cannot but be very useful in a great many cases, and more especially in Annoticism, where we have sometimes occasion to extract even the three hundred sixty-fifth root of a number, as at other times to raise it to the three hundred sixty-fifth power, scarce possible to be performed any other way; to say nothing of the innumerable mistakes that in so long and laborious a calculation would be almost unavoidable, all which are prevented by the use of logarithms. It cannot indeed be expected that entire powers, and much less entire roots, should be gained this way; but it will be easy in most cases to obtain as many terms as can be of any use to us,

7thly, *If any set of numbers, as A, B, C, D, be in continual geometrical proportion, their logarithms, which we shall call a, b, c, d, will be in arithmetical progression:* for since by the supposition A is to B as B is to C as C is to D, that is, since  $\frac{B}{A} = \frac{C}{B} = \frac{D}{C}$ , we shall have  $b - a = c - b = d - c$  by the fourth confectary; therefore a, b, c, d, are in arithmetical progression.  
Q. E. D.

8thly, *From this last confectary it will be easy, having two numbers given, to find as many mean proportionals as we please between them.* Let the given numbers be A and F, and let it be required to find four mean proportionals between them, which we shall call B, C, D, E, so that A, B, C, D, E, F, may be in continual geometrical proportion. Here then it is evident from the last confectary, that, as these numbers

bers are in continual geometrical proportion, their logarithms, which we shall call  $a, b, c, d, e, f$ , will be in arithmetical progression, whereof the extremes  $a$  and  $f$  are known, as being the logarithms of the known numbers  $A$  and  $F$ , and the intermediates may be found thus. Put  $x$  for the common difference of this arithmetic progression; then will  $a+x=b$ ,  $a+2x=c$ ,  $a+3x=d$ ,  $a+4x=e$ ,  $a+5x=f$ ; whence  $x = \frac{f-a}{5}$ ; whence  $a+x$  or  $b = a + \frac{f-a}{5} = \frac{4a+f}{5}$ ,  $a+2x$  or  $c = \frac{3a+2f}{5}$ ,  $a+3x$  or  $d = \frac{2a+3f}{5}$ ,  $a+4x$  or  $e = \frac{a+4f}{5}$ ; so that the logarithms of the four mean proportionals sought are  $\frac{4a+f}{5}, \frac{3a+2f}{5}, \frac{2a+3f}{5}, \frac{a+4f}{5}$ : take then the natural numbers  $B, C, D, E$ , of these logarithms, and they will be the mean proportionals required. Q. E. I.

### *Logarithms the Measures of Ratios.*

391. *Logarithms are so called from their being the arithmetical or numeral exponents of ratios*: for if unity be made the common consequent of all ratios, or the common standard to which all other numbers are to be referred, then every logarithm will be the numeral exponent of the ratio of its natural number to unity. As for instance, the ratio of 81 to 1 actually contains within itself these four ratios, to wit, the ratio of 81 to 27, that of 27 to 9, that of 9 to 3, and that of 3 to 1 (see art. 293); all which ratios are equal to one another, and to the ratio of 3 to 1; therefore the ratio of 81 to 1 is said to be four times as big as the ratio of 3 to 1 (see art. 294): and hence it is that the logarithm of 81 is four times as big as the logarithm of 3. Again, the ratio of 24 to 1 contains, and may be resolved into these three

ratios, to wit, the ratio of 24 to 12, that of 12 to 4, and that of 4 to 1; the first of these ratios, to wit, the ratio of 24 to 12, is the same with that of 2 to 1; the second, to wit, the ratio of 12 to 4, is the same with that of 3 to 1; and therefore the ratio of 24 to 1 is equal to the ratios of 2 to 1, 3 to 1, and 4 to 1, put together; and hence it is that the logarithm of 24 is equal to the logarithms of 2, 3, and 4, put together: *And universally, the magnitude of the ratio of A to 1 is to the magnitude of the ratio of B to 1 as the logarithm of A is to the logarithm of B.* And hence we have a way of measuring all ratios whatever, let their consequents be what they will: as for example, the ratio of A to B is the excess of the ratio of A to 1 above the ratio of B to 1 (see art. 296); therefore the numeral exponent of the ratio of A to B will be the excess of the numeral exponent of the ratio of A to 1 above the numeral exponent of the ratio of B to 1, that is, the excess of the logarithm of A above the logarithm of B; therefore *The magnitude of the ratio of A to B is to the magnitude of the ratio of C to D as the excess of the logarithm of A above the logarithm of B, which is the measure of the former ratio, is to the excess of the logarithm of C above the logarithm of D, which is the measure of the latter ratio:* and thus we see that logarithms are as true and as proper measures of ratios as circular arcs are of angles.

I might have defined logarithms from the idea here given of them, and thence have deduced all the other properties above described: but, as it is not every one that hath a just and distinct notion of the nature and composition of ratios, I thought it more adviseable to treat of them in a way more familiar to the learner.

### Of Briggs's Logarithms.

392. From the definition given in art. 390, it may easily be seen, that, if any one system of logarithms be once obtained, an infinite number of others may be derived from

from them by increasing or diminishing the logarithms of that system in some given proportion. As for instance, in the system given let  $a$ ,  $b$ ,  $c$ , be the logarithms of three numbers,  $A$ ,  $B$ , and  $C$ , whereof the third is the product of the other two multiplied together; then will  $a + b = c$ , by the definition. Let us now imagine all the logarithms of this given system to be doubled; then will  $a$ ,  $b$ , and  $c$ , be changed into  $2a$ ,  $2b$ , and  $2c$ ; but as  $a + b$  was equal to  $c$  in the former system, so now will  $2a + 2b$  be equal to  $2c$  in the latter; that is, all the numbers of this new system will still retain the property of logarithms. But though all these different systems be equally perfect, if computed to the same degree of accuracy, yet they will not all be equally convenient for use: for of all systems or tables of logarithms, that is certainly best accommodated for practice which is now in use, and is commonly known by the name of Briggs's logarithms. The Lord Napier, a Scotch nobleman, was the first inventor of logarithms; but our countryman Mr. Briggs, Professor of Geometry in Gresham College, was undoubtedly the first who thought of this system; and, proposing it to the noble inventor the Lord Napier, he afterwards published it with that Lord's consent and approbation.

The distinguishing mark of this system is, that herein the logarithm of 10 is 1, and consequently that of 100, 2, that of 1000, 3, that of 10000, 4, &c.; that of 1, 0, that of  $\frac{1}{10}$  or of 0.1, - 1. that of  $\frac{1}{100}$  or of 0.01, - 2, &c. In this system the integral parts of the logarithms are always distinguished from the rest, and called the indexes or characteristics of the logarithms whereof they are parts: thus the logarithm of 20 is 1.3010300, where the characteristic is 1; that of 2 is 0.3010300, where the characteristic is 0; that of  $\frac{2}{10}$  or 0.2 is - 1.3010300, where - 1 is the characteristic, &c.

Some

*Some Advantages of this System.*

393. Some of the chief advantages of this system, beyond all others, will appear from the following considerations.

1st, Whereas we have frequent occasion to multiply and divide by 10, 100, 1000, &c. this in this system is very readily performed, only by adding to or subtracting from the characteristic the numbers 1, 2, 3, &c.; and as these are whole numbers, they can only influence the index or characteristic of a logarithm, without affecting the decimal part.

2dly, So long as the digits that compose any number are the same, and in the same order, whatever be their places with respect to the place of units, the decimal parts of the logarithm of such a number will always be the same. As for instance, let  $4 + \frac{1}{10}$  be the logarithm of this number 34567.89, where 4 is the characteristic, and  $\frac{1}{10}$  represents the sum of all the decimal parts; then will  $5 + \frac{1}{10}$  be the logarithm of 345678.9,  $6 + \frac{1}{10}$  that of 3456789,  $7 + \frac{1}{10}$  that of 34567890, &c. On the other hand,  $3 + \frac{1}{10}$  will be the logarithm of 3456.789,  $2 + \frac{1}{10}$  that of 345.6789,  $1 + \frac{1}{10}$  that of 34.56789,  $0 + \frac{1}{10}$  that of 3.456789,  $-1 + \frac{1}{10}$  that of 0.3456789,  $-2 + \frac{1}{10}$  that of 0.03456789, &c.: the reason of this is plain; for if the number 34567.89 be multiplied by 10, the product will be 345678.9; therefore if to  $4 + \frac{1}{10}$ , the logarithm of the former number, be added 1, the logarithm of 10, the sum  $5 + \frac{1}{10}$  will be the logarithm of the latter. Again, if the number 34567.89 be divided by 10, the quotient will be 3456.789; therefore if from  $4 + \frac{1}{10}$ , the logarithm of the former number be subtracted 1, the logarithm of 10, the remainder  $3 + \frac{1}{10}$  will be the logarithm of the latter. Here then we see the reason why in *Briggs's* tables the decimal part of every logarithm is affirmative, whether the whole logarithm taken together be so or not; for, in the logarithm of all numbers greater than unity, both  
the

the integral and decimal parts are affirmative; and therefore the decimal parts must always be so, since these are not changed by changing the natural number, so long as the digits that compose it are the same, and in the same order: thus  $\frac{-3}{10}$  or  $-.3$  may be a logarithm; but it is never expressed so, but rather thus,  $-1+.7$ , the negation being thrown wholly upon the characteristic.

3dly, By this means in *Briggs's* system the characteristic of the logarithm of any number is easily known thus: suppose I was asked, what is the characteristic of the logarithm of this number 34567.89? Here I consider that this number lies between 10000 and 100000; therefore its logarithm must be some number between 4 and 5; therefore it must be 4 with some decimal parts annexed, that is, the characteristic must be 4. And again, suppose it was required to assign the characteristic of the logarithm of this number, 0.03456789:

here I consider that this number lies between  $\frac{1}{10}$  and

$\frac{1}{100}$ , that is, between 0.1 and 0.01, and therefore its logarithm must lie between  $-1$  and  $-2$ , that is, its logarithm must be  $-2$  with some affirmative decimal parts annexed, to lessen the negation; therefore the characteristic will be  $-2$ .

*To find the Characteristic of Briggs's Logarithm of any Number.*

394. Hence may be drawn a short and easy rule for determining the index or characteristic of the logarithm of any number given, thus. *If the number given be a whole number, or a mixt number consisting of integral and decimal parts, then so many removes as is the place of units to the right hand of the first figure, of so many units will the characteristic consist; but if the number proposed be a pure decimal, then so*  
*many*

many removes as is the place of units to the left hand of the first significant figure, of so many negative units will the characteristic consist. Thus the index or characteristic of the logarithm of this number 34567·89 is 4, because 7 in the place of units is four removes to the right hand of the first figure 3: thus again, the characteristic of the logarithm of this number 0·03456789 is -2, because 0 in the place of units is two removes to the left hand of the first significant figure 3.

These rules are the more to be observed, because in some tables the integral parts of all logarithms are omitted, being left to be supplied by the operator himself, as occasion requires: by this means, the logarithms become of much more general use than if, by having their characteristics prefixed, they were tied down to particular numbers.

### *Another Idea of Logarithms.*

395. In the system here described, every natural number is, or may be, considered as some power of 10, and its logarithm as the index of that power: for let  $a$  be the logarithm of any natural number as  $A$ ; then since Briggs's logarithm of 10 is 1, his logarithm of  $10^a$  will be  $a$ ; this is evident from art. 390, sect. 6; therefore  $A$  must be equal to  $10^a$ , since they have both the same logarithm; that is, the natural number  $A$  is such a power of 10 as is expressed by its logarithm  $a$ . This consideration gives us a new idea of logarithms, and to one acquainted with the nature of powers and their indexes, it will be no wonder that the addition, subtraction, multiplication, and division, of these logarithms answer to the multiplication, division, involution, and evolution, of their natural numbers.

### *Precautions to be used in working by Briggs's Logarithms.*

396. Though these logarithms (as I observed before) are preferable to all others, on account of their simplicity

simplicity and facility in practice, yet in using them some precautions are to be observed, which (to prevent mistakes) I shall here just point out to the learner; as,

1st, *In the addition of logarithms, whatever is carried over from the decimal to the integral parts must be considered as affirmative, and as such must be added to those integral parts, whether they be affirmative or negative.* Thus  $-3 + .7000000$  being added to  $-4 + .8000000$ , the sum will be  $-6 + .5000000$ ; for though the sum of the characteristics  $-3$  and  $-4$  be  $-7$ , the affirmative unit drawn from the decimals reduces it to  $-6$ .

2dly, *Whenever a subtraction is to be made in logarithms, it must be performed in the decimal parts as usual; but if the characteristic of the subtrahend, or of the number from whence the subtraction is to be made, or of both, be negative, they must be treated in the subtraction as the nature of such quantities requires.* Thus  $-3 + .8900000$  subtracted from  $-1 + .7600000$  leaves  $1 + .8700000$ : for if  $+1$ , on account of the decimals, be added to  $-3$ , the characteristic of the subtrahend, it will be reduced to  $-2$ , which being subtracted from  $-1$  as above, leaves  $+1$ . Nay, the learner must not be discouraged if he sometimes finds himself obliged to subtract a greater logarithm from a less, as will always be the case where the logarithm of a proper fraction is required: as for example, let it be required to find the logarithm of  $\frac{1}{2}$ : here subtracting  $0 + .3010300$ , the logarithm of  $2$ , from  $0 + .0000000$ , the logarithm of  $1$ , there will remain  $-1 + .6989700$ , the logarithm of  $\frac{1}{2}$ ; for in this subtraction,  $+1$ , on account of the decimals being added to the characteristic of the subtrahend, gives  $1$ , which subtracted from  $0$  above, leaves  $-1$ .

Note, *The logarithm of a vulgar fraction may also be obtained by throwing it into a decimal.* Thus the logarithm of  $\frac{2}{3}$  may be obtained either by subtracting the logarithm of  $3$  from that of  $2$ , or else by taking  
out

out the logarithm of this decimal fraction  $\cdot 6666667$ , which is the same as the logarithm of the whole number  $6666667$ , except that the characteristic of the former logarithm is  $-1$ , and that of the latter  $+6$ .

3dly, *In the multiplication of logarithms the same care must be taken as in addition.* Thus if it be required to multiply this logarithm  $-3 + \cdot 7000000$  by 9; the product will be  $-21 + \cdot 3000000$ ; for though the product of  $-3 \times 9$  be  $-27$ , yet the  $+6$  drawn from the decimals reduces it to  $-21$ .

4thly, *Whenever a logarithm is to be divided by 2, 3, 4, &c. in order to obtain the square, cube, biquadrate, &c. root of its natural number, if the characteristic be negative, and will not be divided without a fraction, my way is to resolve it into two parts, to wit, into a negative part which will be divided, and an affirmative part which will incorporate with the decimals annexed.* Thus if I was to take the half of this logarithm  $-1 + \cdot 7000000$ , I cannot join the  $-1$  to the decimals annexed, because they are quantities of different kinds; therefore I resolve the characteristic  $-1$  into two parts, to wit,  $-2 + 1$ , and then taking the half of  $-2$ , which is  $-1$ , I join the affirmative part  $+1$  to the decimals annexed, and so take the half of  $+17$ , which is  $+8$ , &c.; therefore the half of the aforesaid logarithm is  $-1 + \cdot 8500000$ : had the characteristic been  $-3$ , I should have resolved it into  $-4 + 1$ . Had  $\frac{1}{3}$  of the foresaid logarithm been required, I should have resolved the characteristic  $-1$  into  $-3 + 2$ , and so should have taken, first, the third part of  $-3$ , which is  $-1$ , and then of  $+27$ , which is  $+9$ : had the characteristic been  $-2$ , I should have resolved it into  $-3 + 1$ ; had it been  $-4$ , I should have resolved it into  $-6 + 2$ , and so on.

*N. B.* Of all the tables hitherto in use whose logarithms do not run to above seven decimal places, I take those published by Doctor Sherwin to be the best upon many accounts, and particularly in the disposition of the logarithms: these therefore I shall not scruple

scruple to recommend to my readers, whom I shall also refer to the directions there given for finding the logarithms of all absolute numbers from 1 to 10000000, and *vice versa*. But I must own I cannot with equal justice recommend the method there taken to avoid negative indexes by creating new ones, and by using arithmetical complements. It is not to be denied but that this sort of practice may be absolutely necessary to such as know nothing of the nature and use of negative quantities; but those who do, I believe, will find the rules here laid down more natural and convenient; and as they carry their own reasons along with them, I doubt not but that the learner will find them easier to be remembered, and less liable to be misunderstood.

397. In the tables above recommended, after the logarithms on every page, are two columns, one called a column of differences, and signed *D*; the other called a column of proportional parts, and signed *Pts* above, and *Pro* below: these two columns, as well as the rest, have been explained by the author; but, lest they should not be thoroughly understood by what is there said of them, I shall take the liberty, by a single instance, to explain more at large the reason and use of these columns: I shall take my example from the author himself. Let it then be required to find by the tables the logarithms of this number of seven places, to wit, 5423758: to do this, I first put down 6, the characteristic of the logarithm sought, according to the directions given in art. 394; then I consider in the next place, that though by the help of the tables we can find the logarithm of any number under 10000000, yet that the absolute numbers there do not, properly speaking, run to above five places; therefore I lower the absolute number given, to wit, 5423758, to this, 54237.58, which will not affect the decimal part of the logarithm sought; then, setting aside the characteristic, I take out of the tables the logarithm of the five integral places 54237 according

according to the directions there given, and find it to be 7342957; this I subtract from the logarithm of 54238, that is, from 7343037, and find the difference to be 80. But the design of the column of differences is on purpose to avoid this subtraction: for, had I taken out of that column the number opposite to 54237, the integral part of the absolute number proposed, or if no such opposite number was to be found, had I taken the nearest number above (not below), I should have found the number 80.1, that is, in a whole number, 80, without any subtraction. Thus then the case stands: as the absolute number proposed 54237.58 lies between the two nearest tabular numbers 54237 and 54238, whose difference is 1, so must the logarithm sought lie between the logarithms of the tabular numbers above mentioned, whose difference is 80; therefore I say by the golden rule, as 1, the difference of the two tabular numbers, between which mine lies, is to 80, the difference of the two tabular logarithms between which the logarithm sought lies, so is .58, the difference betwixt my number and the nearest less tabular number, to 46, the difference betwixt the logarithm sought, and the nearest less tabular logarithm; therefore adding this difference 46 to the nearest less tabular logarithm, to wit, 7342957, I have 7343003, which being joined as decimal parts to the characteristic 6, gives 6.7343003 for the logarithm sought. This number 46, which was the fourth proportional above found, is called the proportional part, because it is the same proportional part of 80, the difference of the two nearest tabular logarithms, that .58, the decimal part of the number proposed, is of 1, the difference of the two nearest tabular numbers. Whoever attends to the foregoing operation will easily perceive, that this proportional part 46 was gained from multiplying 80, the common difference, by .58, the decimal part of the absolute number proposed; and the same would have been obtained if the common difference

80 had first been multiplied by  $\cdot 5$  and then by  $\cdot 08$ , and the products been taken into one sum: now it is to save these two multiplications that the column of proportional parts was contrived; for whoever looks there for the common difference 80 will find all the products of the said common difference multiplied by  $\cdot 1$ ,  $\cdot 2$ ,  $\cdot 3$ ,  $\cdot 4$ ,  $\cdot 5$ , &c. to  $\cdot 9$  inclusively; and looking for the number over against  $\cdot 5$ , he will find the number 40, which shews that the number 40 is  $\frac{5}{10}$  of the common difference 80; so also over against 8 he will find the number 64, which shews that the number 64 is  $\frac{8}{10}$  of the common difference; but we do not want  $\frac{8}{10}$  of it, but 8 hundredth parts; therefore he must not take the number 64, but a tenth part of that number, to wit,  $6 \cdot 4$  or 6, which being added to 40, the proportional part before found, gives 46, to be added to the nearest less tabular logarithm in order to obtain the logarithm sought.

But when all possible exactness is required, and no errors are intended to be committed, but such as unavoidably arise from the imperfection of the logarithms themselves; I would advise the reader to compute the proportional parts himself, as above, rather than trust to the table for them, though he will rarely find any considerable difference. My reason for this advice is, because in the table of proportional parts, no notice is taken of decimals; whereas those decimals ought not in all cases to be neglected, at least not till the operation is over, and the artist sees what it is he throws away or takes into his account, to lessen the error as much as he can.

*Note.* For the construction of logarithms see the Quarto Edition, pp. 627 to 640. The same subject is also largely treated of in my Mathematical Essays. J. H.

F I N I S.



The following are the names of the  
persons who have been admitted to the  
membership of the Society since the  
last meeting. The names are given in  
the order in which they were admitted.  
The names of the persons who have  
been admitted to the membership of the  
Society since the last meeting are given  
in the following list.













